Special Relativity with Four-Vectors

This is no introduction to the Special Theory of Relativity. The reader should be familiar with the main concepts of that theory as presented in any standard text book on that topic. A concise introduction called 'Fast Track' can be found on https://www.physastromath.ch/uploads/myPdfs/Relativ/Fast Track.pdf .
We will oftly refere to results proven in that "Fast Track". So [1 - 21.3] points to formula 21.3 and [1 - 3] to section 3 of that paper.
In addition, the reader should have some knowledge of matrix calculations.

The aim of this paper is to introduce four-vectors as a powerful tool to solve problems in STR. The sections **A1** to **A11** give the complete theory of four-vectors. A general proof is given of the frame-independence of the defined scalar product.
Sections **B12** to **B25** show how to solve problems in STR using those four-vectors or the 'half speed' introduced in the 'Fast Track'.
Sections **C26** to **C40** deal with electro-magnetism and the Lorentz force law. Four-vectors are great to handle electro-magnetism in STR. In **C39** we can prove the form-invariance of Maxwell's equations by a change of the inertial frame by some simple matrix-multiplications.

Useful sources:

[1] 'Fast Track' , Martin Gubler, Januar 2021 (link see above)

[2] 'Relativitätstheorie für Studienanfänger', Jürgen Freund, vdf ETH Zürich 2005[2]

[3] 'Raum Zeit Relativität', Roman Sexl und Herbert K. Schmidt, vieweg studium 36, 1993[3]

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A1 Four-Position X

The Lorentz transformations show how to calculate the coordinates (t',x',y',z') of an event in a reference frame S' from the coordinates (t,x,y,z) of that event in the reference frame S and vice versa - supposed the reference frames S and S' are in a special relationship: The x-axis and the x'-axis fall into one, and both of the y-axes and z-axes are parallel to each other. Furthermore, the relative speed v of the reference frames is parallel to the x-axis: $\vec{\boldsymbol{v}} = (v, 0, 0)$ as seen from S. Finally, the primary clocks in both systems, sitting at the origins of the frames, have both been resetted to zero at the moment the origins of the systems met (see [1 - 20]). All the other clocks in the systems S and S' have been synchronized with the primary clock of their frame.

The Lorentz transformations are given by the following set of equations [see 1 - 20.5] :

$$\begin{aligned} t &= \gamma_v \cdot \left(t' + \beta_v \cdot \frac{x'}{c}\right) & \qquad t' &= \gamma_v \cdot \left(t - \beta_v \cdot \frac{x}{c}\right) \\ x &= \gamma_v \cdot (x' + \beta_v \cdot c \cdot t') & x' &= \gamma_v \cdot (x - \beta_v \cdot c \cdot t) \\ y &= y' & y' &= y \\ z &= z' & z' &= z \end{aligned} \tag{1.1}$$

γ and β are the well-known abbreviations $\gamma = \gamma_v = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$ and $\beta = \beta_v = \frac{v}{c}$. (1.2)

Using the Lorentz matrix L the equations (1.1) can be expressed by a single matrix equation:

$$\begin{pmatrix} c \cdot t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma \cdot \beta & 0 & 0 \\ -\gamma \cdot \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c \cdot t \\ x \\ y \\ z \end{pmatrix} \tag{1.3}$$

Time is multiplied by the speed of light in order to have the same units for all four components of the vector. The vector containing all four coordinates of an event is called the *four-position* of that event. X and X' are frequently used as symbols for four-positions. Using these symbols and the name L for the Lorentz-matrix we can rewrite (1.3) as

$$X' = L \cdot X \tag{1.4}$$

The inverse transformation is done using the inverse Matrix L^{-1}:

$$X = L^{-1} \cdot X' \tag{1.5}$$

L^{-1} is almost identical to L , you just have to drop the minus signs.

Vectors that transform according to (1.4) when the inertial frame changes from S to S' are generally called *four-vectors*.

Four-position X is oftly written as $X = (c \cdot t\,,\ \vec{x}\,)$. $c \cdot t$ is the temporal part, the 3d-vector $\vec{x}$ is the spatial part of the four-vector. A four-vector is always thought as a matrix with one column and four rows.

A2 Linear Combinations of Four-Vectors

Multiplication by any matrix is a linear mapping of vectors. Therefore, if X and Y are four-vectors, any linear combination $m \cdot X + n \cdot Y$ of X and Y is a four-vector, too, if m and n have the same value in all reference frames.

Hence, the difference $X(t_2) - X(t_1)$ of two four-positions is another four-vector: $\Delta X = (c \cdot \Delta t\, ,\ \Delta\vec{x}\,)$

Rest mass m_0 and rest charge density ϱ_0 are invariant. We will use those constants to create important four-vectors based on the four-velocity U: Four-momentum P is defined by $P = m_0 \cdot U$ and the four-current J by $J = \varrho_0 \cdot U$.

A3 Differentiating with Respect to Proper Time, Four-Velocity U

Differentiating the four-position $X = (c \cdot t\,,\ \vec{x}\,)$ with respect to time t yields the vector $(c\,,\ \vec{u}\,)$. $\vec{u}$ is the ordinary 3d-velocity of some object in system S. But $(c\,,\ \vec{u}\,)$ does not qualify as a four-vector. The first component does not transform according to (1.3):

$$\gamma \cdot c - \gamma \cdot \beta \cdot u_x \neq c$$

Insert $\vec{u} = (0, c/2, c/2)$, for example.

Deriving with respect to time t cannot result in a four-vector, as time runs at a different speed in each coordinate frame. But there is one distinguished coordinate frame with any moving object: The actual rest frame of that moving object, the so-called *eigen system* or *the comoving inertal frame*. For all frames the following equation holds, where τ is the time in the comoving inertial frame :

$$\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \Delta\tau^2 - 0 - 0 - 0 = \Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 \tag{3.1}$$

The *proper time interval* $\Delta\tau$ is a relativistic invariant. A proof of (3.1) is given by theorem (8.4) of that paper. Hence, following **A2** , the vector $\frac{1}{\Delta\tau} \cdot \Delta X$ is a four-vector, too.
This holds for arbitrarily small time intervals $\Delta\tau$, and so it is true in the limit of $\Delta\tau \to 0$. Deriving the four-position with respect to proper time gives us the four-velocity U :

$$U = \lim_{\Delta\tau \to 0} \frac{\Delta X}{\Delta\tau} = \frac{d}{d\tau}(X) \tag{3.2}$$

(3.1) clearly shows that Δt is greater than $\Delta\tau$. Hence we have

$$\Delta\tau = \Delta t \cdot \sqrt{1 - \frac{u^2}{c^2}} = \Delta t \,/\, \gamma_u \qquad \text{and} \qquad \frac{dt}{d\tau} = \gamma_u \tag{3.3}$$

Using the chain rule we can calculate the derivative with respect to τ :

$$U = \frac{d}{d\tau}(X) = \frac{d}{dt}(X) \cdot \frac{dt}{d\tau} = \gamma_u \cdot \frac{d}{dt}(X) = \gamma_u \cdot \frac{d}{dt}(c \cdot t\,, \vec{x}\,) = \gamma_u \cdot (c\,, \vec{u}\,) \tag{3.4}$$

$U = \gamma_u \cdot (c\,, \vec{u}\,)$ is the four-velocity we were looking for.

The derivation of any four-vector with respect to proper time τ produces another four-vector, and we know now how to calculate this derivation.

The four-velocity U is sort of an artificial construct. It can not be measured, all we can measure is the traditional 3d-velocity. But, as we will show, it is a great tool to do calculations in STR.

The *proper velocity* of some object in its *eigen frame* is always $U = \gamma_0 \cdot (c\,, \vec{0}\,) = 1 \cdot (c\,, \vec{0}\,) = (c, 0,0,0)^T$ (3.5)

The superscript character T indicates the *transposition* of the 1x4-row-matrix to a 4x1-column-matrix or a vector.

A4 Four-Momentum P

Following **A2** , multiplying four-velocity U by the rest mass m_0 we get another four-vector. That four-vector is called four-momentum :

$$P = m_0 \cdot U = m_0 \cdot \gamma_u \cdot (c , \vec{u}) = (\tfrac{1}{c} \cdot E_{tot} , \vec{p}) \quad (4.1)$$

3d-vector $\vec{p} = \gamma_u \cdot m_0 \cdot \vec{u}$ is the SRT momentum vector, and for the temporal part we have used the equation $E_{tot} = \gamma_u \cdot m_0 \cdot c^2$.

A5 Four-Current J

Multiplying four-velocity U by the charge density ρ_0 measured in the rest system of the charge we get the four-current

$$J = \rho_0 \cdot U = \rho_0 \cdot \gamma_u \cdot (c , \vec{u}) \quad (5.1)$$

This representation is perfect for a cloud of charged particels who move all with velocity $\vec{u}$ in the same direction.
In a wire, only the electrons in the conduction band are moving, and total charge density in the wire is zero. Only a small part of all charged particels contribute to the current. There we need the more general expression

$$J = (\rho \cdot c , \vec{j}) \quad (5.2)$$

for the four-current. ρ is the charge density measured in the actual reference frame, and $\vec{j}$ is the 3d-vector of current-density. $j_x \cdot A_x$ is the current I_x in x-direction, when A_x stands for the cross section of the wire orthogonal to the x-axis and $j_x = \rho \cdot u_x$ gives the current density in that x-direction (→ **C32**).

A6 Four-Force K

Following **A3** the derivation of a four-vector with respect to proper time gives another four-vector. Deriving the four-momentum P with respect to proper time we get the four-force K :

$$K = \frac{d}{d\tau}(P) = \frac{d}{dt}(P) \cdot \frac{dt}{d\tau} = \gamma \cdot \frac{d}{dt}(P) = \gamma \cdot \frac{d}{dt}(\tfrac{1}{c} \cdot E , \vec{p}) = \gamma \cdot (\tfrac{1}{c} \cdot \frac{dE}{dt} , \frac{d\vec{p}}{dt}) \quad (6.1)$$

By definition we have $\frac{d\vec{p}}{dt} = \vec{f}$, with $\vec{f}$ denoting the ordinary 3d force vector. And for $\frac{dE}{dt}$ we have by definition

$$\frac{dE}{dt} = \vec{f} \cdot \vec{u} = \vec{f} \cdot \frac{d\vec{x}}{dt} \quad \text{or} \quad dE = \vec{f} \cdot d\vec{x} \quad (6.2)$$

So the four-force K can be written as

$$K = \frac{d}{d\tau}(P) = \gamma \cdot (\tfrac{1}{c} \cdot \frac{dE}{dt} , \frac{d\vec{p}}{dt}) = \gamma \cdot (\tfrac{1}{c} \cdot \vec{f} \cdot \vec{u} , \vec{f}) \quad (6.3)$$

where E stands for the total energy E_{tot} . The character F will be used later to denote the Faraday matrix specifying the electromagnetic field.

A7 Four-Acceleration A

The derivation of the four-velocity U with respect to proper time τ gives us the four-acceleration

$$A = \frac{d}{d\tau}(U) = \frac{d}{dt}(U)\cdot\frac{dt}{d\tau} = \gamma\cdot\frac{d}{dt}(U) = \gamma\cdot\frac{d}{dt}(\gamma\cdot(c,\vec{u})) = \gamma\cdot\left(\frac{d}{dt}(\gamma)\cdot(c,\vec{u}) + \gamma\cdot\frac{d}{dt}(c,\vec{u})\right) \tag{7.1}$$

So we have to derive γ with respect to t :

$$\frac{d}{dt}(\gamma) = \frac{d}{dt}\left[\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}\right] = \frac{d}{d\vec{u}}\left[\left(1-\frac{\vec{u}^2}{c^2}\right)^{-\frac{1}{2}}\right]\cdot\frac{d\vec{u}}{dt} = -\frac{1}{2}\cdot\left[\left(1-\frac{\vec{u}^2}{c^2}\right)^{-\frac{3}{2}}\right]\cdot\left(-\frac{2\,\vec{u}}{c^2}\right)\cdot\vec{a} = \gamma^3\cdot c^{-2}\cdot\vec{u}\cdot\vec{a} \tag{7.2}$$

$\vec{a} = \frac{d\vec{u}}{dt}$ is the 3d acceleration vector. So (7.1) can be developped to

$$A = \frac{d}{d\tau}(U) = \gamma\cdot\left(\frac{d}{dt}(\gamma)\cdot(c,\vec{u}) + \gamma\cdot\frac{d}{dt}(c,\vec{u})\right) = \gamma^4\cdot c^{-2}\cdot\vec{u}\cdot\vec{a}\cdot(c,\vec{u}) + \gamma^2\cdot(0,\vec{a}) \tag{7.3}$$

By definition in SRT the equation $K = m_0\cdot A$ is always true :

$$K = \frac{d}{d\tau}(P) = \frac{d}{d\tau}(m_0\cdot U) = m_0\cdot\frac{d}{d\tau}(U) = m_0\cdot A \tag{7.4}$$

Combining (6.3) with (7.3) we get

$$K = \gamma\cdot\left(\frac{1}{c}\cdot\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right) = \gamma\cdot\left(\frac{1}{c}\cdot\vec{f}\cdot\vec{u}, \vec{f}\right) = m_0\cdot[\gamma^4\cdot c^{-2}\cdot\vec{u}\cdot\vec{a}\cdot(c,\vec{u}) + \gamma^2\cdot(0,\vec{a})] \tag{7.5}$$

A close inspection of (7.5) shows that force $\vec{f}$ and acceleration $\vec{a}$ do not need to be parallel in STR !

(7.5) shows that if $\vec{u}$ and $\vec{a}$ are parallel then $\vec{f}$ and $\vec{a}$ have to be parallel, too. In that case the temporal part of (7.5) says

$$\frac{dE}{dt} = \vec{f}\cdot\vec{u} = \gamma^3\cdot m_0\cdot\vec{u}\cdot\vec{a}$$

and we have

$$\vec{f} = \gamma^3\cdot m_0\cdot\vec{a} \tag{7.6}$$

Hence the term 'longitudinal mass' for $\gamma^3\cdot m_0$ used around 1905 in papers on that topic.

If force $\vec{f}$ and velocity $\vec{u}$ are perpendicular to each other the first summand on the right side of (7.3) and (7.5) is zero. Then from (7.5) we get the relation

$$\vec{f} = \gamma\cdot m_0\cdot\vec{a} \tag{7.7}$$

The term $\gamma\cdot m_0$ was called 'transversal mass'. In that case the rate of change of energy $\frac{dE}{dt}$ is zero. Think of a charged particle moving in a magnetic field, when the only force acting on the particle is the Lorentz force.

A8 A Special Inner Product for Four-Vectors

The power of four-vectors as a tool for calculations in STR comes from a special inner product defined for four-vectors. The result of this inner product is independent of the reference frame used to calculate the product, it is relativistically invariant. So we can use in any situation the reference frame that makes the calculation as simple as possible.

Let X^i and Y^i be two four-vectors with components x_0 to x_3 and y_0 to y_3 . We define the inner product $\circ$ by

$$X^i \circ Y^i \equiv x_0 \cdot y_0 - x_1 \cdot y_1 - x_2 \cdot y_2 - x_3 \cdot y_3 \tag{8.1}$$

Obvously that inner product is commutative.

Let us look at two examples.

Let $X^i = (c \cdot t\,,\ \vec{x}\,) = (c \cdot t, x, y, z)^T$ be the four-position of some particle. Definition (8.1) says

$$X^i \circ X^i = (c \cdot t)^2 - x^2 - y^2 - z^2 = (c \cdot \tau)^2 \tag{8.2}$$

Indeed, the result does not depend of the reference frame used to calculate it.

Let us do the same with four-velocity $U^i = \gamma \cdot (c\,, \vec{u}\,) = \gamma \cdot (c, u_x, u_y, u_z)^T$. We calculate $U^i \circ U^i$:

$$U^i \circ U^i = \gamma \cdot (c\,, -\vec{u}\,)^T \cdot \gamma \cdot (c\,, \vec{u}\,) = \gamma^2 \cdot (c^2 - \vec{u}^2) = \frac{1}{1 - \frac{u^2}{c^2}} \cdot (c^2 - u^2) = \frac{c^2}{c^2 - u^2} \cdot (c^2 - u^2) = c^2 \tag{8.3}$$

Again, the result does not depend of the choice of the reference frame.

Theorem: The value of the inner product $X \circ Y$ of four-vectors does not depend of the reference frame used to do the calculation: $X \circ Y = X' \circ Y'$ (8.4)

In order to give a proof of the theorem we will introduce a further mathematical tool in the next section: So called four-forms.

A9 Four-Forms and the Inner Product

To each four-vector $X^i = (x_0, x_1, x_2, x_3)^T$ we define a corresponding four-form by

$$X_i = (x_0, -x_1, -x_2, -x_3) \tag{9.1}$$

A four-form is a matrix with one row and four columns, while a four-vector is represented by a matrix with one column and four rows. Be aware or the position of the index i !

Using four-forms we can write the inner product of section **A8** as an ordinary product of matrices:

$$X^i \circ Y^i = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \circ \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \equiv x_0 \cdot y_0 - x_1 \cdot y_1 - x_2 \cdot y_2 - x_3 \cdot y_3 = (x_0, -x_1, -x_2, -x_3) \cdot \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = X_i \cdot Y^i \tag{9.2}$$

By the symmetry of our inner product we have

$$X_i \cdot Y^i = X^i \circ Y^i = Y^i \circ X^i = Y_i \cdot X^i \tag{9.3}$$

The last piece we need to give a very short proof of theorem (8.4) is the matrix G :

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{9.4}$$

Simple calculations show that the following equations hold:

$$G^T = G = G^{-1} \quad , \quad L^{-1} = G^{-1} \cdot L \cdot G \quad , \quad L = G^{-1} \cdot L^{-1} \cdot G \quad , \quad X_i = (G \cdot X^i)^T = (X^i)^T \cdot G \tag{9.5}$$

Four-vectors are defined to obey $X^{i\prime} = L \cdot X^i$. But how to transform the corresponding four-forms ?
Using (9.5) we get from $X^{i\prime} = L \cdot X^i$

$$X_i' = (X^{i\prime})^T \cdot G = (L \cdot X^i)^T \cdot G = (X^i)^T \cdot L^T \cdot G = (X^i)^T \cdot L \cdot G =$$

$$= (X^i)^T \cdot (G \cdot G^{-1}) \cdot L \cdot G = [(X^i)^T \cdot G] \cdot [G^{-1} \cdot L \cdot G\,] = X_i \cdot L^{-1}$$

So four-forms are transformed from one reference frame to another by

$$X_i{}' = X_i \cdot L^{-1} \qquad \text{and} \qquad X_i = X_i{}' \cdot L \tag{9.6}$$

Now we are well prepared for the short proof of theorem (8.4) :

$$X^{i\prime} \circ Y^{i\prime} = X_i{}' \cdot Y^{i\prime} = (X_i \cdot L^{-1}) \cdot (L \cdot Y^i) = X_i \cdot (L^{-1} \cdot L) \cdot Y^i = X_i \cdot Y^i = X^i \circ Y^i \tag{9.7}$$

q.e.d.

In section **C37** we will introduce an important four-form in order to prepare the proof of another great theorem.

A10 Some Selected Products of Four-Vectors

In section **A8** we have calculated $U \circ U$. Now we will use theorem (8.4) to do that calculation. A particle with four-velocity $U = \gamma \cdot (c, \vec{u})$ has the proper velocity $U' = 1 \cdot (c, \vec{0})$, and so we have

$$U \circ U = U' \circ U' = 1 \cdot (c, -\vec{0})^T \cdot 1 \cdot (c, \vec{0}) = c^2 \tag{10.1}$$

In the eigen system of the particle the calculation is absolutely simple. Compare with the calculation in **A8** .

Using (10.1) we find for the inner product of the four-momentum $P = m_0 \cdot U$ and the four-velocity U

$$P \circ U = (m_0 \cdot U) \circ U = m_0 \cdot (U \circ U) = m_0 \cdot c^2 = E_0 \tag{10.2}$$

and

$$P \circ P = (m_0 \cdot U) \circ (m_0 \cdot U) = {m_0}^2 \cdot (U \circ U) = {m_0}^2 \cdot c^2 \tag{10.3}$$

Now let $A = \frac{d}{d\tau}(U)$ be the four-acceleration of a particle with with the four-velocity U . In the eigen system of the moving particle we have $\vec{u}' = 0$ and $U' = 1 \cdot (c, \vec{0})$. From that we get with (7.3) $A' = \gamma^2 \cdot (0, \vec{a}') = 1 \cdot (0, \vec{a}')$. So we can calculate the inner product of A and U using theorem (8.4) :

$$A \circ U = A' \circ U' = (0, -\vec{a}')^T \cdot (c, \vec{0}) = (0, \vec{0}) = 0 \tag{10.4}$$

Hence we have for the four-force

$$K \circ U = (m_0 \cdot A) \circ U = m_0 \cdot (A \circ U) = m_0 \cdot 0 = 0 \tag{10.5}$$

For the four-acceleration A we have according to (7.3)

$$A = \gamma^4 \cdot c^{-2} \cdot \vec{u} \cdot \vec{a} \cdot (c, \vec{u}) + \gamma^2 \cdot (0, \vec{a})$$

The components are given by

$$\begin{aligned} A^0 &= \gamma^4 \cdot c^{-2} \cdot \vec{u} \cdot \vec{a} \cdot c \\ A^1 &= \gamma^4 \cdot c^{-2} \cdot \vec{u} \cdot \vec{a} \cdot u_x + \gamma^2 \cdot a_x \\ A^2 &= \gamma^4 \cdot c^{-2} \cdot \vec{u} \cdot \vec{a} \cdot u_y + \gamma^2 \cdot a_y \\ A^3 &= \gamma^4 \cdot c^{-2} \cdot \vec{u} \cdot \vec{a} \cdot u_z + \gamma^2 \cdot a_z \end{aligned}$$

and hence

$$A \circ A = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2 = \gamma^8 \cdot c^{-4} \cdot (\vec{u} \cdot \vec{a})^2 \cdot \left[c^2 - {u_x}^2 - {u_y}^2 - {u_z}^2\right] -$$

$$2 \cdot \gamma^6 \cdot c^{-2} \cdot (\vec{u} \cdot \vec{a}) \cdot \left[u_x \cdot a_x + u_y \cdot a_y + u_z \cdot a_z\right] - \gamma^4 \cdot \left[{a_x}^2 + {a_y}^2 + {a_z}^2\right] =$$

$$= \gamma^8 \cdot c^{-4} \cdot (\vec{u} \cdot \vec{a})^2 \cdot [c^2 - u^2] - 2 \cdot \gamma^6 \cdot c^{-2} \cdot (\vec{u} \cdot \vec{a})^2 - \gamma^4 \cdot a^2 =$$

$$= \gamma^8 \cdot c^{-2} \cdot (\vec{u} \cdot \vec{a})^2 \cdot \frac{c^2 - u^2}{c^2} - 2 \cdot \gamma^6 \cdot c^{-2} \cdot (\vec{u} \cdot \vec{a})^2 - \gamma^4 \cdot a^2 =$$

$$= \gamma^6 \cdot c^{-2} \cdot (\vec{u} \cdot \vec{a})^2 - 2 \cdot \gamma^6 \cdot c^{-2} \cdot (\vec{u} \cdot \vec{a})^2 - \gamma^4 \cdot a^2 = -\gamma^6 \cdot c^{-2} \cdot (\vec{u} \cdot \vec{a})^2 - \gamma^4 \cdot a^2$$

The general result ist

$$A \circ A = -\gamma^6 \cdot c^{-2} \cdot (\vec{u} \cdot \vec{a})^2 - \gamma^4 \cdot a^2 \tag{10.6}$$

In the eigen system of the accelerated particle we have $\vec{u}' = 0$ and $\gamma = 1$. There, (10.6) reduces to $A' \circ A' = -a'^2 \equiv -\alpha^2$. The *proper acceleration* gets the symbol $\vec{\alpha}$ (alpha). So (10.6) can be expanded to

$$A \circ A = -\gamma^6 \cdot c^{-2} \cdot (\vec{u} \cdot \vec{a})^2 - \gamma^4 \cdot a^2 = -\alpha^2 \qquad (10.7)$$

If $\vec{u}$ and $\vec{a}$ are perpendicular to each other (e.g. in a storage ring with a Lorentz force at work) (10.7) reduces to

$$\vec{\alpha} = \gamma^2 \cdot \vec{a} \qquad \text{and} \qquad \alpha = \gamma^2 \cdot a = \gamma^2 \cdot \frac{u^2}{r} \qquad (10.8)$$

In a linear accelerator $\vec{u}$ and $\vec{a}$ are parallel to each other. Then (10.7) reduces to

$$\alpha^2 = \gamma^6 \cdot c^{-2} \cdot (\vec{u} \cdot \vec{a})^2 + \gamma^4 \cdot a^2 = \gamma^6 \cdot c^{-2} \cdot u^2 \cdot a^2 + \gamma^4 \cdot a^2 =$$

$$= \gamma^4 \cdot a^2 \cdot \left(\gamma^2 \cdot \frac{u^2}{c^2} + 1\right) = \gamma^4 \cdot a^2 \cdot \left(\frac{c^2}{c^2-u^2} \cdot \frac{u^2}{c^2} + 1\right) =$$

$$= \gamma^4 \cdot a^2 \cdot \left(\frac{u^2}{c^2-u^2} + \frac{c^2-u^2}{c^2-u^2}\right) = \gamma^4 \cdot a^2 \cdot \left(\frac{c^2}{c^2-u^2}\right) =$$

$$= \gamma^4 \cdot a^2 \cdot \gamma^2 = \gamma^6 \cdot a^2$$

In that case the proper acceleration $\vec{\alpha}$ is

$$\vec{\alpha} = \gamma^3 \cdot \vec{a} \qquad (10.9)$$

and (7.6) tells us

$$\vec{f} = \gamma^3 \cdot m_0 \cdot \vec{a} = \boldsymbol{m_0} \cdot \vec{\boldsymbol{\alpha}} \qquad (10.10)$$

If a centripetal force is at work we have, following (7.7) and (10.8)

$$\vec{f} = \gamma \cdot m_0 \cdot \vec{a} = \frac{\mathbf{1}}{\gamma} \cdot \boldsymbol{m_0} \cdot \vec{\boldsymbol{\alpha}} \qquad (10.11)$$

A11 Four-Momentum as a Conserved Quantity

The conservation laws for momentum and energy merge into one to the conservation law for four-momentum.

Conservation of four-momentum means

$$\sum_i P_i \; = \; \sum_j P_j$$

where the sum runs over all particles involved in a collision bevor (i) and after (j) that collision.

In **A4** we noticed

$$P_i \; = \; (\frac{1}{c} \cdot E_i \, , \vec{p}_i \,)$$

The sum over the first components means total energy divided by the speed of light. So the conservation of the first component of the four-momentum guarantees conservation of total energy.
The sum over the spatial components of four-momentum means total 3d momentum. The conservation of that sum guarantees conservation of total 3d momentum. Be aware that $\vec{p} \; = \gamma_u \cdot m_0 \cdot \vec{u}$ is the SRT momentum vector.

The second part of this paper brings a lot of examples, many of them illustrating the power of calculations with four-vectors. Starting from conservation of four-momentum we will build inner products with selected four-vectors to eliminate unknown variables. This may look as follows :

$$P_1 \, + \, P_2 \, = \, P_3 \, + \, P_4 \qquad \Longrightarrow \qquad P_1 \circ P_4 + \, P_2 \circ P_4 \; = \; P_3 \circ P_4 \, + \, P_4 \circ P_4$$

or

$$P_1 \, + \, P_2 \, = \, P_3 \, + \, P_4 \qquad \Longrightarrow \qquad (P_1 + \, P_2) \circ (P_1 + \, P_2) \; = \; (P_3 \, + \, P_4) \circ (P_3 + \, P_4)$$

For each inner product we are free then to choose the reference frame to do the calculation.

B12 Energy, Momentum and Rest Energy

As a theoretical application of four-vectors we give a proof of the well known equation

$$E_{tot}{}^{2} = E_{0}{}^{2} + p^2 \cdot c^2 \tag{12.1}$$

(12.1) is true in every frame of reference.

Let P denote the four-momentum of some object in the reference frame S. We have $P = (E_{tot}/c\,,\ \vec{p}\,)$. In the rest frame of that object, i.e. in its co-moving inertial frame, the four-momentum of that object is $P_0 = 1 \cdot m_0 \cdot (c\,,\ \vec{0}\,) = (E_0/c\,,\ \vec{0}\,)$. Now we calculate the inner products $P \circ P$ and $P_0 \circ P_0$ and use theorem (8.4) :

- $P \circ P = (E_{tot}/c)^2 - p^2$ by the definition of the inner product
- $P_0 \circ P_0 = (E_0/c)^2 - 0$ by the definition of the inner product
- both terms are equal, and hence $E_{tot}{}^{2} - p^2 \cdot c^2 = c^2 \cdot (P \circ P) = c^2 \cdot (P_0 \circ P_0) = E_0{}^{2}$ q.e.d.

B13 The Four-Momentum of Light Quanta

For light quanta alias photons with $m_0 = 0$ (12.1) reduces to $E_{tot}{}^{2} = 0 + p^2 \cdot c^2$. So we have

$$E_{tot} = p \cdot c = E_{kin} = E \tag{13.1}$$

For particles with speed c we have $p = E/c$. The four-vector of a photon looks like

$$P = (E/c\,,\vec{p}\,) = \frac{E}{c} \cdot (1\,,\vec{1}\,) = \frac{h \cdot f}{c} \cdot (1\,,\vec{1}\,) \tag{13.2}$$

If the photon runs in y-direction the unit vector $\vec{1}$ can be written as $\vec{1} = (0,1,0)$.

For the four-momentum of light quanta we always have

$$P \circ P = \frac{h \cdot f}{c} \cdot \frac{h \cdot f}{c} \cdot (1 - 1) = 0 \tag{13.3}$$

In general we have $P \circ P = (E_0/c)^2$. For light particles with zero rest mass (they are really light ...) we get from that (13.3) too.

B14 The Fast Observers Measurements

Let some object move with four-momentum $P = (E_{tot}/c \, , \, \vec{p} \,) \; = \; \gamma \cdot m_0 \cdot (c \, , \vec{v} \,)$ in the reference frame S. For somebody resting in S (observer A) we have

- $E_{tot} \; = \; \gamma \cdot m_0 \cdot c^2 \; = \; c \cdot P^0 \; = \; U_0 \circ P$ with the proper velocity $U_0 \; = \; 1 \cdot (\, c \, , \vec{0} \,)$
- $E_0 \; = \; c \cdot \sqrt{P \circ P}$ because of $P \circ P \; = \; {m_0}^2 \cdot c^2$
- $E_{kin} = \; E_{tot} - E_0 \; = \; U_0 \circ P \; - \; c \cdot \sqrt{P \circ P}$
- $m_0 \; = \; \sqrt{P \circ P} \; / \, c$

Now let the observer B move with velocity U in the frame S . That observer observes the same object as before in his reference frame S'. What are the values observer B ascribes to the moving object ?

- E_0 and m_0 have the same values for B as they had for A. These values are *invariant*.
- ${E_{tot}}' = {U_0}' \circ P' \; = \; U \circ P$ where ${U_0}' = \; 1 \cdot (\, c \, , \vec{0} \,)$ denotes the proper velocity of B in his frame S'
- ${E_{kin}}' \; = \; {E_{tot}}' - {E_0}' \; = \; U \circ P \; - \; c \cdot \sqrt{P \circ P}$

All of that values are easily calculated in both frames of reference.

What if the moving object is a photon ist ? Then we have $P = (E_{tot}/c \, , \, \vec{p} \,) \; = \; \frac{h \cdot f}{c} \cdot \, (\, 1 \, , \vec{1} \,)$ with some unit vector $\vec{1}$.

For both observers we have $m_0 = 0$ and $E_0 \; = 0$, and for both $E \; = \; E_{kin} = \; E_{tot}$ holds true. But the energy of the photon differs for A and B :

For A we have $E \; = \; U_0 \circ P \; = \; 1 \cdot (\, c \, , \vec{0} \,) \circ \frac{h \cdot f}{c} \cdot \, (\, 1 \, , \vec{1} \,) \; = \; c \cdot P^0 \; = \; h \cdot f$

For B we have $E' \; = \; U \circ P \; = \; {U_0}' \circ P' \; = \; 1 \cdot (\, c \, , \vec{0} \,) \circ \frac{h \cdot f'}{c} \cdot \, (\, 1 \, , \vec{1} \,) \; = \; h \cdot f'$

in both cases the energy can be calculated with the inner product of four-vectors.

B15 Pair Annihilation 1

Let us look at a head-on collision of a electron and a positron. The particle and its anti-particle disappear and a pair of quanta carries away momentum and energy. We write down the four-momenta of all particles in the center of mass system of the incoming leptons :

A the four-momentum of the electron : $A = \gamma \cdot m_0 \cdot (c , \vec{v})$

B the four-momentum of the positron : $B = \gamma \cdot m_0 \cdot (c , -\vec{v})$

In the center of mass system total 3d momentum is zero before the collision. So total momentum has to be zero after the collision, too. Hence the necessity of **two** photons heading away after the collision with equal energies in opposite directions :

C the four-momentum of one of the photons : $C = \frac{h \cdot f}{c} \cdot (1 , \vec{1})$

D the four-momentum of the other photon : $D = \frac{h \cdot f}{c} \cdot (1 , -\vec{1})$

The argument above is based on the conservation of the spatial components of four-momentum

$$A + B = C + D$$

The temporal component, i.e. conservation of energy, yields

$$2 \cdot \gamma \cdot m_0 \cdot c = 2 \cdot h \cdot f / c$$

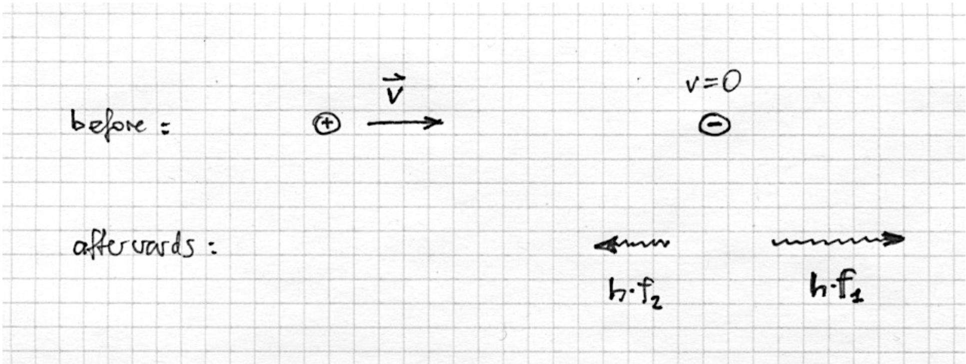
and hence

$$h \cdot f = \gamma \cdot m_0 \cdot c^2$$

The direction of flight of the photons is unknown.

B16 Pair Annihilation 2

Now let a fast positron hit an electron at rest in some reference frame S. We know from **B15** that two quanta are created. Their 3d momenta carry on the 3d momentum of the incoming positron. In this section we are going to calculate the energies (or the frequencies) of the quanta in a special case: Both quanta should move along the line of the incoming positron :



We do that calculation in a reference frame T that moves with 'half of the speed' of v in the direction of v (for that 'half speed' w consult [1 - 3]). In that reference frame T we are back in the situation of **B15** ! Both quanta have the same frequency f' with

$$h \cdot f' = \gamma_w \cdot m_0 \cdot c^2 = m_0 \cdot c^2 \cdot \left(1 - \frac{w}{c}\right)^{-\frac{1}{2}} \cdot \left(1 + \frac{w}{c}\right)^{-\frac{1}{2}} \tag{16.1}$$

Using formula [1 - 1.4] for the longitudinal Doppler shift we can calculate the corresponding frequencies in frame S :

$$h \cdot f_1 = h \cdot f' \cdot \sqrt{\frac{c+w}{c-w}} = m_0 \cdot c^2 \cdot \left(1 - \frac{w}{c}\right)^{-\frac{1}{2}} \cdot \left(1 + \frac{w}{c}\right)^{-\frac{1}{2}} \cdot \left(1 + \frac{w}{c}\right)^{+\frac{1}{2}} \cdot \left(1 - \frac{w}{c}\right)^{-\frac{1}{2}} = m_0 \cdot c^2 \cdot \frac{c}{c-w}$$

and (16.2)

$$h \cdot f_2 = h \cdot f' \cdot \sqrt{\frac{c-w}{c+w}} = m_0 \cdot c^2 \cdot \left(1 - \frac{w}{c}\right)^{-\frac{1}{2}} \cdot \left(1 + \frac{w}{c}\right)^{-\frac{1}{2}} \cdot \left(1 - \frac{w}{c}\right)^{+\frac{1}{2}} \cdot \left(1 + \frac{w}{c}\right)^{-\frac{1}{2}} = m_0 \cdot c^2 \cdot \frac{c}{c+w}$$

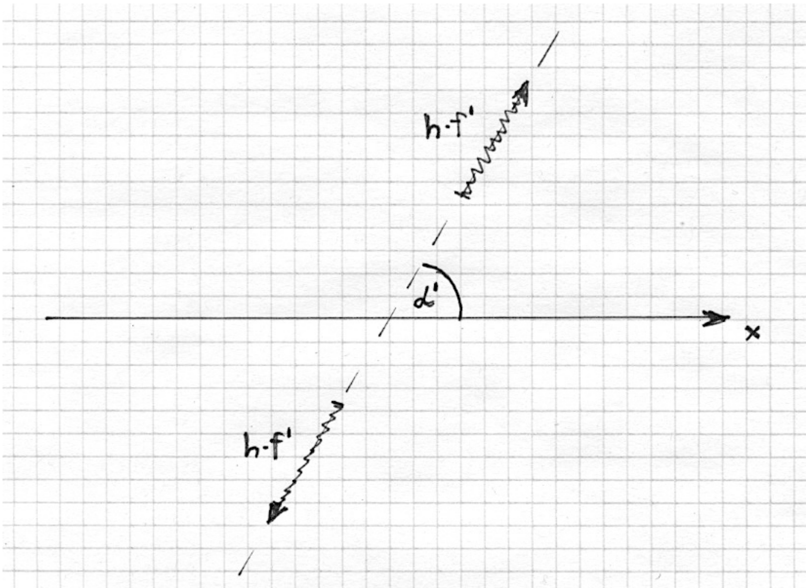
It is not easy to show that energy conservation is guaranteed with (16.2). I am thankful to Mathematica® to do that calculation for me ... So the following equation is correct :

$$h \cdot f_1 + h \cdot f_2 = \gamma_v \cdot m_0 \cdot c^2 + m_0 \cdot c^2$$

[2 - 29.44] gives solutions to this problems that look rather complicated. By means of the 'half speed' w many problems can be solved with a minimal mathematical effort.

B17 Pair Annihilation 3

We are back again in the situation of **B16**. But now, the quantas are allowed to fly off in any direction in frame T :



System T is moving to the right with speed w as seen from S . So the upper quant moves towards the observer resting in S and hence has the increased frequency $f_1 > f'$ in frame S, while the quant below has its frequency Doppler shifted to $f_2 < f_1$. We use the general Doppler formula [1 - 22.1] :

$$f_S = f_T \cdot \frac{1}{\gamma_w \cdot (1 - \frac{w}{c} \cdot cos\varphi)}$$

For f_T we have to insert f' from section **B16** . We get the increased frequency f_1 if we insert α' , and we get the lower frequency f_2 if we insert $180° - \alpha'$ for φ :

$$h \cdot f_1 = h \cdot f' \cdot \frac{1}{\gamma_w \cdot (1 - \frac{w}{c} \cdot cos(\alpha'))} = \gamma_w \cdot m_0 \cdot c^2 \cdot \frac{1}{\gamma_w \cdot (1 - \frac{w}{c} \cdot cos(\alpha'))} = \frac{m_0 \cdot c^2}{1 - \frac{w}{c} \cdot cos(\alpha')}$$

$$h \cdot f_2 = h \cdot f' \cdot \frac{1}{\gamma_w \cdot (1 - \frac{w}{c} \cdot cos(180° - \alpha'))} = \gamma_w \cdot m_0 \cdot c^2 \cdot \frac{1}{\gamma_w \cdot (1 + \frac{w}{c} \cdot cos(\alpha'))} = \frac{m_0 \cdot c^2}{1 + \frac{w}{c} \cdot cos(\alpha')}$$

With $\alpha' = 0°$ and $cos(\alpha') = 1$ we get the results of the last section **B16** .

Again, using the 'half speed' w simplifies the calculation and gives a quite handsome result.

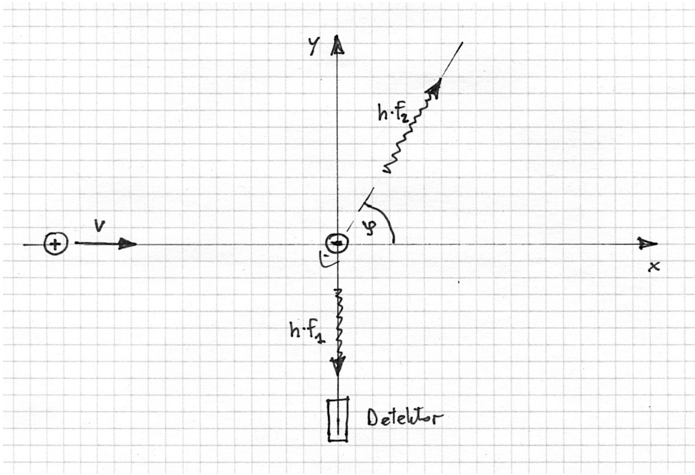
In system S both of the angles α and β are smaller compared to α' resp. $180° - \alpha'$. α and β can be calculated with the aberration formula [1-22.3] :

$$tan\,\frac{\alpha}{2} = \sqrt{\frac{c-w}{c+w}} \cdot tan\,\frac{\alpha'}{2} \quad \text{and} \quad tan\,\frac{\beta}{2} = \sqrt{\frac{c-w}{c+w}} \cdot tan\,\frac{180°-\alpha'}{2}$$

The 3d momenta of the quanta have to catch the 3d momentum of the incoming positron.

B18 Pair Annihilation 4

One more time we are back in the situation of **B16**. Now a detector catches only quanta that fly off at a right angle to the direction of the incoming positron :



We will calculate the energies of both of the quanta and the angle φ in the figure above. Let us list the four-momenta of all of the involved particles :

- $P_1 = \gamma_v \cdot m_0 \cdot (c, v, 0, 0)$ the four-momentum of the incoming positron
- $P_2 = m_0 \cdot (c, 0, 0, 0)$ the four-momentum of the electron at rest in S
- $P_3 = \frac{h}{c} \cdot f_1 \cdot (1, 0, -1, 0)$ the four-momentum of the quantum catched by the detector
- $P_4 = \frac{h}{c} \cdot f_2 \cdot (1, cos(\varphi), sin(\varphi), 0)$ the four-momentum of the other quantum

Conservation of four-momentum means $P_1 + P_2 = P_3 + P_4$. The first three components of that vector equation yield three equations for the unknown variables f_1 , f_2 and φ :

- $\gamma_v \cdot m_0 \cdot c + m_0 \cdot c = \frac{h}{c} \cdot f_1 + \frac{h}{c} \cdot f_2$
  multiplied by c we get (no surprise) $E_1 + E_2 = E_3 + E_4$ (18.1)
- $\gamma_v \cdot m_0 \cdot v + 0 = 0 + \frac{h}{c} \cdot f_2 \cdot cos(\varphi)$
  multiplied by c we get $E_1 \cdot \frac{v}{c} = E_4 \cdot cos(\varphi)$ (18.2)
- $0 = \frac{h}{c} \cdot f_2 \cdot (-1) + \frac{h}{c} \cdot f_2 \cdot sin(\varphi)$
  multiplied by c we get $E_3 = E_4 \cdot sin(\varphi)$ (18.3)

In a first step we get rid off the angle φ by adding the squares of the equations (18.2) and (18.3) :

$$E_1{}^2 \cdot \frac{v^2}{c^2} + E_3{}^2 = E_4{}^2 \cdot (sin^2(\varphi) + cos^2(\varphi)) = E_4{}^2 \quad \text{or} \quad E_4{}^2 - E_3{}^2 = E_1{}^2 \cdot \frac{v^2}{c^2} \tag{18.4}$$

Now we multiply (18.1) by $E_4 - E_3$ and use (18.4) to get

$$(E_1 + E_2) \cdot (E_4 - E_3) = (E_3 + E_4) \cdot (E_4 - E_3) = E_4{}^2 - E_3{}^2 = E_1{}^2 \cdot \frac{v^2}{c^2} \tag{18.5}$$

Dividing (18.5) by $(E_1 + E_2)$ we get $$E_4 - E_3 = \frac{E_1{}^2 \cdot \frac{v^2}{c^2}}{E_1 + E_2} \quad (18.6)$$

(18.1) still says $$E_4 + E_3 = E_1 + E_2 \quad (18.7)$$

Adding (18.6) and (18.7) we find $$2 \cdot E_4 = E_1 + E_2 + \frac{E_1{}^2 \cdot \frac{v^2}{c^2}}{E_1 + E_2} \quad (18.8)$$

$\frac{v^2}{c^2}$ can be expressed by the energies E_1 and E_2 as follows :

$$1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2} = \frac{E_2{}^2}{E_1{}^2} \qquad \text{and hence} \qquad \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} = 1 - \frac{E_2{}^2}{E_1{}^2} = \frac{E_1{}^2 - E_2{}^2}{E_1{}^2} \quad (18.9)$$

Inserting (18.9) in (18.8) we get

$$2 \cdot E_4 = E_1 + E_2 + \frac{E_1{}^2}{E_1 + E_2} \cdot \frac{E_1{}^2 - E_2{}^2}{E_1{}^2} = E_1 + E_2 + E_1 - E_2 = 2 \cdot E_1 \quad (18.10)$$

Together with (18.7) we find finally $$E_4 = E_1 \quad \text{and} \quad E_3 = E_2 \quad (18.11)$$

The angle φ can be calculated by

$$\cos(\varphi) = \frac{p_{1x}}{p_4} = \frac{\gamma \cdot m_0 \cdot v}{E_4/c} = \frac{\gamma \cdot m_0 \cdot v}{E_1/c} = \frac{\gamma \cdot m_0 \cdot v}{\gamma \cdot m_0 \cdot c} = \frac{v}{c} = \sqrt{1 - \frac{E_2{}^2}{E_1{}^2}} \quad (18.12)$$

All solutions are given in terms using the energies E_1 and E_2 only. The solutions (18.11) and (18.12) are rather simpel, and they clearly satisfy the equations (18.1) to (18.3). Let me write down the solutions once again :

- $h \cdot f_1 = E_3 = E_2 = m_0 \cdot c^2$
- $h \cdot f_2 = E_4 = E_1 = \gamma_v \cdot m_0 \cdot c^2$
- $\cos(\varphi) = \frac{v}{c} = \sqrt{1 - \frac{E_2{}^2}{E_1{}^2}} = \sqrt{1 - \frac{1}{\gamma^2}}$

Measuring f_1 and φ allows to calculate the energy of the incoming positron.

B19 Pair Creation

A high energy quantum can not decay into an electron-positron-pair without the presence of another particle. In the center of mass frame of the created particles total momentum would be zero, while the momentum of the incoming quantum is not zero in any frame of reference. Only the presence of another particle, usually the kernel of an atom, allows that process to take place. Good luck for the astronomers: The quanta are forbidden to decay spontaneously in empty space, most of them travel unchanged over 'astronomical' distances.

In the rest frame of the involved kernel we have

- $P_1 = \frac{h \cdot f}{c} \cdot (1, 1, 0, 0)$ the four-momentum of the incoming quant
- $P_2 = (M \cdot c, 0, 0, 0)$ the four-momentum of the kernel
- P_3 the four-momentum of the **cluster** containing the new particles **and** the kernel after that the pair creation took place

(it is impossible to calculate the single momenta of all of the three particles after the pair-creation without further informations).

Conservation of four-momentum means

$$P_1 + P_2 = P_3$$

The square of this equation is

$$P_1 \circ P_1 + 2 \cdot P_1 \circ P_2 + P_2 \circ P_2 = P_3 \circ P_3 \tag{19.1}$$

$P_1 \circ P_1$ equals zero, $P_2 \circ P_2$ equals $(M \cdot c)^2$ and $P_1 \circ P_2$ results in $h \cdot f \cdot M$. We calculate the square of P_3 in the rest frame of the cluster. There we have (the kernel being much heavier than the created leptons)

$$P_3' \approx \left((M + 2 \cdot m_0) \cdot c, 0, 0, 0\right) \qquad \text{and hence} \qquad P_3 \circ P_3 = P_3' \circ P_3' \approx (M + 2 \cdot m_0)^2 \cdot c^2 \,.$$

Inserting these terms in (19.1) we get

$$0 + 2 \cdot h \cdot f \cdot M + (M \cdot c)^2 \approx (M + 2 \cdot m_0)^2 \cdot c^2$$

expanded

$$2 \cdot h \cdot f \cdot M + M^2 \cdot c^2 \approx M^2 \cdot c^2 + 4 \cdot M \cdot m_0 \cdot c^2 + 4 \cdot m_0{}^2 \cdot c^2$$

and simplified

$$h \cdot f \approx 2 \cdot m_0 \cdot c^2 + 2 \cdot m_0{}^2 \cdot c^2 / M = 2 \cdot m_0 \cdot c^2 \cdot \left(1 + \frac{m_0}{M}\right) \tag{19.2}$$

The result shows again that without the presence of that kernel, i.e. with $M = 0$, the input energy would go to infinity, and the pair creation could not take place. If, instead of a kernel, the involved particle is an electron the energy of the incoming quant has to be at least twice the rest energy of the created particles.

B20 The Perfectly Inelastic Collision

Let two particles with rest mass m_a and m_b move along the x-direction in system S with velocities $\vec{u}_a = (u_a, 0, 0)$ and $\vec{u}_b = (u_b, 0, 0)$. After a completely inelastic collision they build a single new particle. We want to calculate the rest mass m_c and the velocity $\vec{u}_c = (u_c, 0, 0)$ of that new particle in system S.

Conservation of four-momentum means $P_a + P_b = P_c$. Squared we have

$$P_a \circ P_a + 2 \cdot P_a \circ P_b + P_b \circ P_b = P_c \circ P_c$$

The square terms are calculated in the rest frame of the particle. So we get

$$m_a{}^2 \cdot c^2 + 2 \cdot \gamma_a \cdot m_a \cdot (c, u_a, 0, 0) \circ \gamma_b \cdot m_b \cdot (c, u_b, 0, 0) + m_b{}^2 \cdot c^2 = m_c{}^2 \cdot c^2$$

Dividing by c^2 and calculating $P_a \circ P_b$ we find

$$m_a{}^2 + 2 \cdot \gamma_a \cdot m_a \cdot \gamma_b \cdot m_b \cdot \frac{(c^2 - u_a \cdot u_b)}{c^2} + m_b{}^2 = m_c{}^2$$

rearranged

$$m_c{}^2 = m_a{}^2 + m_b{}^2 + 2 \cdot m_a \cdot m_b \cdot \left(\gamma_a \cdot \gamma_b \cdot \left(1 - \frac{u_a \cdot u_b}{c^2}\right)\right) \tag{20.1}$$

Let us compare (20.1) with

$$(m_a + m_b)^2 = m_a{}^2 + m_b{}^2 + 2 \cdot m_a \cdot m_b$$

In (20.1) we have the additional factor

$$k = \gamma_a \cdot \gamma_b \cdot \left(1 - \frac{u_a \cdot u_b}{c^2}\right) = \left(1 - \frac{u_a{}^2}{c^2}\right)^{-\frac{1}{2}} \cdot \left(1 - \frac{u_b{}^2}{c^2}\right)^{-\frac{1}{2}} \cdot \left(1 - \frac{u_a \cdot u_b}{c^2}\right)$$

If the signs u_a and u_b differ, i.e. if the particles collide with opposite velocities, all three factors of k are greater than 1 and m_c is greater than $m_a + m_b$. If one of the velocities equals zero the third factor disappears, one of the first two factors equals 1 and the other is greater than 1. With some algebra we could show that k is greater than 1 in the last case too, where u_a and u_b have the same sign. The rest mass of the new particle is always greater than the sum of the rest masses of the colliding particles. There is always some part of kinetic energy of the colliding particles that is converted into rest mass of the new particle.

What about the velocity u_c of the new particle ?
Conservation of energy, i.e. the temporal part of conservation of four-momentum, says

$$\gamma_a \cdot m_a \cdot c^2 + \gamma_b \cdot m_b \cdot c^2 = \gamma_c \cdot m_c \cdot c^2 \tag{20.2}$$

Conservation of 3d momentum, i.e. the spatial part of conservation of four-momentum, yields

$$\gamma_a \cdot m_a \cdot u_a + \gamma_b \cdot m_b \cdot u_b = \gamma_c \cdot m_c \cdot u_c \tag{20.3}$$

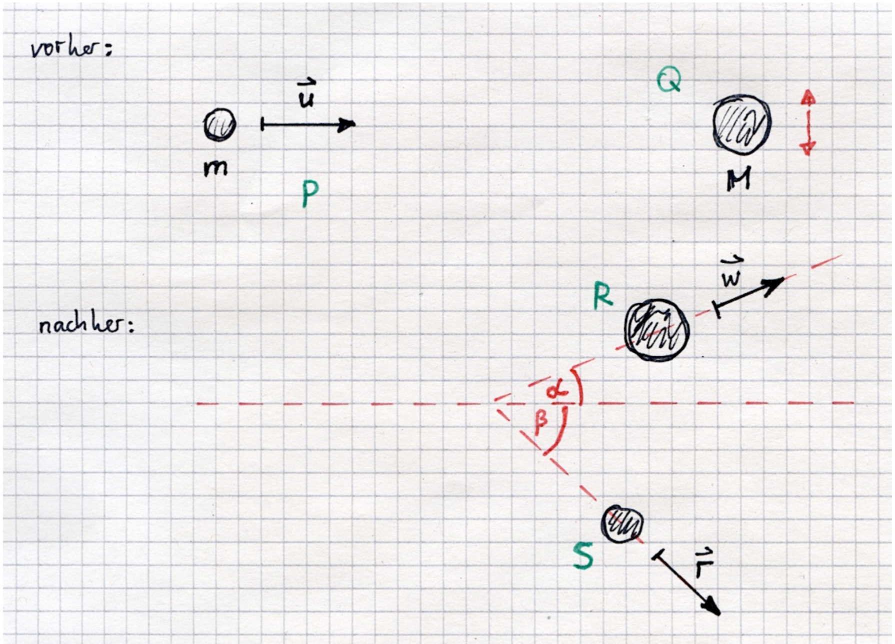
Dividing (20.2) by c^2 we get an expression for $\gamma_c \cdot m_c$. Dividing (20.3) by that term we get

$$u_c = \frac{\gamma_a \cdot m_a \cdot u_a + \gamma_b \cdot m_b \cdot u_b}{\gamma_a \cdot m_a + \gamma_b \cdot m_b} \tag{20.4}$$

u_c is the velocity of the center of mass of the particles - before and after the collision.

B21 The Perfectly Elastic Collision 1

Let a particle with rest mass m and velocity $u = u_x$ collide with a particle with mass M at rest. The collision does not need to be head-on :



Given the velocity u , the ratio m/M of the masses and the angle α between $\vec{w}$ and the x-axis we will calculate the velocities of the particles after the collision.

Let us start with the conservation of four-momentum : $\quad P + Q = R + S \quad$ where (21.1)

$P = \gamma_u \cdot m \cdot (c, u, 0,0)$ for the pushing particle with mass m before the collision
$Q = M \cdot (c, 0,0,0)$ for the resting particle with mass M before the collision
$R = \gamma_w \cdot M \cdot (c, w \cdot cos(\alpha), w \cdot sin(\alpha),0)$ for the pushed particle after the collision
$S = \gamma_r \cdot m \cdot (c, r \cdot cos(\alpha), r \cdot sin(\alpha),0)$ for the pushing particle after the collision

Squaring (21.1) we get $\quad P \circ P + 2 \cdot P \circ Q + Q \circ Q = R \circ R + 2 \cdot R \circ S + S \circ S \quad$ (21.2)

From $P \circ P = S \circ S$ and $Q \circ Q = R \circ R$ follows $P \circ Q = R \circ S$. Now we multiply (21.1) by R :

$$P \circ R + Q \circ R = R \circ R + S \circ R = R \circ R + P \circ Q \qquad (21.3)$$

We have eliminated S and can calculate w now. Inserting the terms

$P \circ R = \gamma_u \cdot \gamma_w \cdot m \cdot M \cdot (c^2 - u \cdot w \cdot cos(\alpha))$, $Q \circ R = \gamma_w \cdot M^2 \cdot c^2$, $R \circ R = M^2 \cdot c^2$ and $P \circ Q = \gamma_u \cdot m \cdot M \cdot c^2$

in (21.3) we get a linear equation for w with the solution

$$w = \frac{2 \cdot \left(1 + \frac{M}{m} \cdot \frac{1}{\gamma_u}\right) \cdot u \cdot \cos(\alpha)}{\left(1 + \frac{M}{m} \cdot \frac{1}{\gamma_u}\right)^2 + u^2 \cdot cos^2(\alpha)} \qquad (21.4)$$

From w and α we get $\quad w_x = w \cdot cos(\alpha)$, $w_y = w \cdot sin(\alpha)$ and $\gamma_w = (1 - w^2)^{-\frac{1}{2}} \quad$ (21.5)

The first component of (21.1), i.e. the conservation of energy, says $\gamma_u \cdot m \cdot c + M \cdot c = \gamma_w \cdot M \cdot c + \gamma_r \cdot m \cdot c$
Solving for γ_r we find

$$\gamma_r = \gamma_u + \frac{M}{m} - \frac{M}{m} \cdot \gamma_w \qquad \text{and hence} \qquad r = \sqrt{1 - \frac{1}{{\gamma_r}^2}} \tag{21.6}$$

From conservation of momentum, i.e. from the second and the third component of (21.1), we get equations for the components of velocity r and the angle β :

$$\gamma_u \cdot m \cdot u = \gamma_r \cdot m \cdot r_x + \gamma_w \cdot M \cdot w_x \quad \rightarrow \quad r_x = \left(\gamma_u \cdot u - \gamma_w \cdot \frac{M}{m} \cdot w_x\right) / \gamma_r$$

$$0 = \gamma_r \cdot m \cdot r_y + \gamma_w \cdot M \cdot w_y \quad \rightarrow \quad r_y = \left(-\gamma_w \cdot \frac{M}{m} \cdot w_y\right) / \gamma_r \tag{21.7}$$

By choice w_y has a positive sign, and hence the sign of r_y is negative while the sign of r_x can have either value.

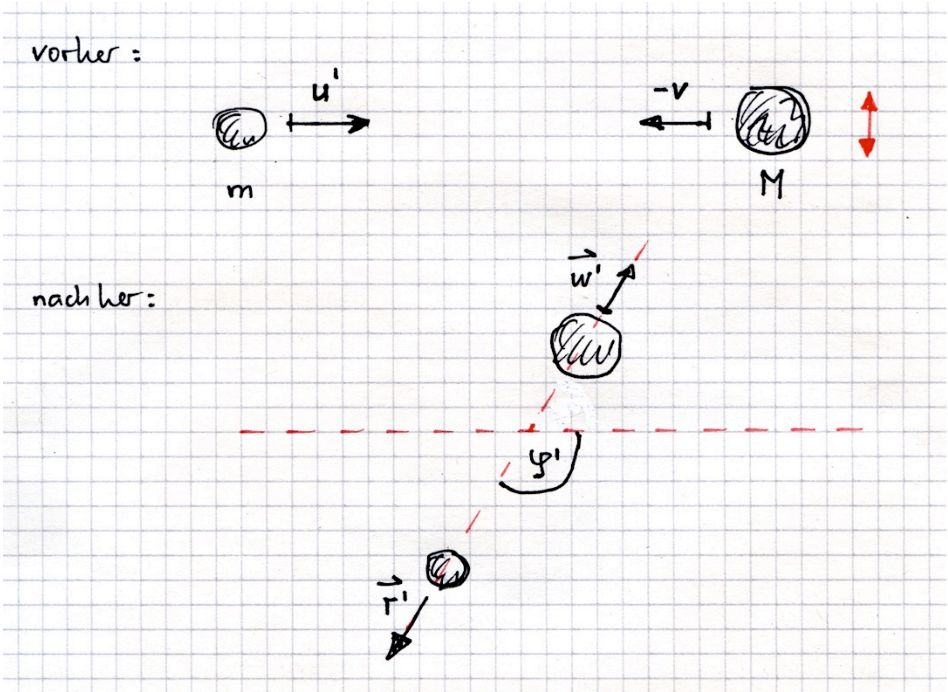
Finally, β is given by

$$\beta = sin^{-1}(r_x/r) \tag{21.8}$$

If you like to play with different values of u , M/m and α you can do so using my GeoGebra program written for this situation. The input parameters are adjusted with sliders, and the program shows visually and numerically the result of the collision : https://www.physastromath.ch/uploads/myPdfs/GeoGebra/ElastStoss_1.ggb

B22 The Perfectly Elastic Collision 2

This section presents another solution for the problem of **B21** (look at the figure at the beginning of **B21**). This time we do the calculations in the center of mass system S' of the two particles :



Let v be the velocity of S' as seen from S. The particle M , resting in S, moves in S' with velocity $-v$ in the direction of the x-axis. In S', total momentum is zero before and after the collision. The particles necessarily have to move in opposite directions after the collision. Conservation of momentum and conservation of energy imply

$$|w'| = |-v| \quad \text{und} \quad |r'| = |u'| \tag{22.1}$$

The velocity v of S' as seen from S can be calculated with formula [1 - 7.1] :

$$v = \frac{p_{tot} \cdot c^2}{E_{tot}} = \frac{\gamma_u \cdot m \cdot u \cdot c^2}{\gamma_u \cdot m \cdot c^2 + M \cdot c^2} = \frac{\gamma_u}{\gamma_u + M/m} \cdot u \tag{22.2}$$

So we know v and γ_v .

To be able to calculate the velocities w' and r' after the collision we need to know u , M/m and the angle φ'.
φ' is the angle between $\overrightarrow{r'}$ and the positive x-axis. φ' may be obtuse if $m < M$.

Following (22.1) we have $\quad r' = u' \;,\; r_x{}' = u' \cdot cos(\varphi') \quad$ and $\quad r_y{}' = -u' \cdot sin(\varphi')$ (22.3)

and $\quad w' = v \;,\; w_x{}' = -v \cdot cos(\varphi') \quad$ and $\quad w_y{}' = v \cdot sin(\varphi')$ (22.4)

These are the results in system S'. With formulas [1 - 22.1] and [1 - 22.2] we will calculate the corresponding velocities in system S.

We find

$$r_x = \frac{r_x{}' + v}{1 + v \cdot r_x{}'/c^2} \quad , \quad r_y = \frac{r_y{}'}{\gamma_v \cdot (1 + v \cdot r_x{}'/c^2)} \quad \text{and} \quad r = \sqrt{r_x{}^2 + r_y{}^2} \tag{22.5}$$

and

$$w_x = \frac{w_x{}' + v}{1 + v \cdot w_x{}'/c^2} \quad , \quad w_y = \frac{w_y{}'}{\gamma_v \cdot (1 + v \cdot w_x{}'/c^2)} \quad \text{and} \quad w = \sqrt{w_x{}^2 + w_y{}^2} \tag{22.6}$$

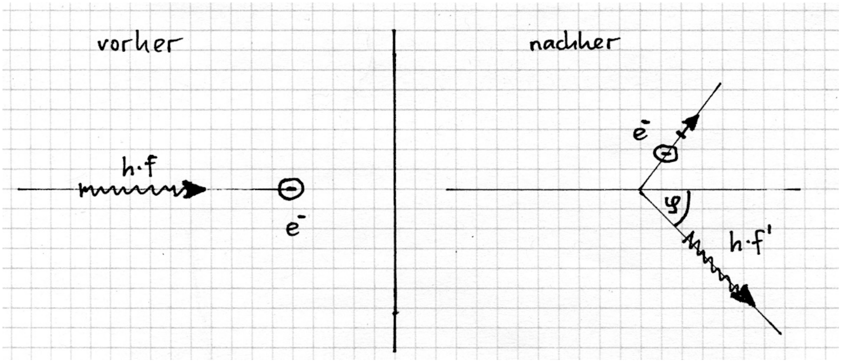
Finally we have to calculate the angles α and β between the velocities and the x-axis in System S. This can be done in many ways. Our choice is

$$\alpha = tan^{-1}(w_y/w_x) \ = \ cos^{-1}(w_x/w) \quad \text{and} \quad \beta = cos^{-1}(r_x/r) \tag{22.7}$$

I have written another GeoGebra program to match exactly the situation in this section. You can adjust u , M/m and φ' with sliders and then watch the result of the corresponding collision.
The link to this little program is https://www.physastromath.ch/uploads/myPdfs/GeoGebra/ElastStoss_2.ggb

B23 Compton Scattering

Let us study the elastic collision of a photon with a free electron at rest :



Before the collision we have the four-momenta $P = \frac{h \cdot f}{c} \cdot (1, 1, 0, 0)$ and $Q = m_0 \cdot (c, 0, 0, 0)$

After the collision we have $R = \frac{h \cdot f'}{c} \cdot (1, \cos\varphi, \sin\varphi, 0)$ and $Q' = \gamma_u \cdot m_0 \cdot (c, \vec{u})$

The starting point is, as usual, the conservation of four-momentum $P + Q = R + Q'$ (23.1)

Squared $P \circ P + 2 \cdot Q \circ P + Q \circ Q = R \circ R + 2 \cdot Q' \circ R + Q' \circ Q'$

and hence $0 + 2 \cdot Q \circ P + Q \circ Q = 0 + 2 \cdot Q' \circ R + Q \circ Q$ and $Q \circ P = Q' \circ R$ (23.2)

Multiplying (23.1) by P' and inserting (23.2) we find $P \circ R + Q \circ R = R \circ R + Q' \circ R = 0 + Q \circ P$ (23.3)

We got rid off Q' ! The remaining inner products of (23.3) are

- $P \circ R = \frac{h \cdot f}{c} \cdot \frac{h \cdot f'}{c} \cdot (1 - \cos\varphi) = \frac{h}{\lambda} \cdot \frac{h}{\lambda'} \cdot (1 - \cos\varphi) = \frac{1}{c^2} \cdot E \cdot E' \cdot (1 - \cos\varphi)$
- $Q \circ R = m_0 \cdot c \cdot \frac{h \cdot f'}{c} = m_0 \cdot c \cdot \frac{h}{\lambda'} = \frac{1}{c^2} \cdot m_0 \cdot c^2 \cdot E'$
- $Q \circ P = m_0 \cdot c \cdot \frac{h \cdot f}{c} = m_0 \cdot c \cdot \frac{h}{\lambda} = \frac{1}{c^2} \cdot m_0 \cdot c^2 \cdot E$

Inserting these terms in (23.3) $\frac{h}{\lambda} \cdot \frac{h}{\lambda'} \cdot (1 - \cos\varphi) + m_0 \cdot c \cdot \frac{h}{\lambda'} = m_0 \cdot c \cdot \frac{h}{\lambda}$

multiplying by $\frac{\lambda \cdot \lambda'}{h}$ $h \cdot (1 - \cos\varphi) + m_0 \cdot c \cdot \lambda = m_0 \cdot c \cdot \lambda'$

and rearranging the terms $h \cdot (1 - \cos\varphi) = m_0 \cdot c \cdot (\lambda' - \lambda)$

we find the Compton scattering formula $\lambda' - \lambda = \frac{h}{m_0 \cdot c} \cdot (1 - \cos\varphi)$ (23.4)

$\frac{h}{m_0 \cdot c} \approx 2.426$ picometer is called the *Compton Wavelength* of the electron.

On the next page we calculate the energy $E' = h \cdot f'$ of the scattered photon in terms of the energy of the incoming photon and the scattering angle φ .

Let us rewrite (23.3) using the energy terms instead of wavelengths :

$$\frac{1}{c^2} \cdot E \cdot E' \cdot (1 - \cos\varphi) \;+\; \frac{1}{c^2} \cdot m_0 \cdot c^2 \cdot E' \;=\; \frac{1}{c^2} \cdot m_0 \cdot c^2 \cdot E \qquad (23.5)$$

Multiplyied by c^2 and slightly rearranged

$$E' \cdot (E \cdot (1 - \cos\varphi) \;+\; m_0 \cdot c^2) \;=\; m_0 \cdot c^2 \cdot E$$

and solved for E' we find

$$E' \;=\; \frac{m_0 \cdot c^2 \cdot E}{E \cdot (1 - \cos\varphi) \;+\; m_0 \cdot c^2} \;=\; E \cdot \frac{1}{1 \;+\; \frac{E}{m_0 \cdot c^2} \cdot (1 - \cos\varphi)} \qquad (23.6)$$

or, equivalently

$$f' \;=\; \frac{m_0 \cdot c^2 \cdot h \cdot f}{h \cdot f \cdot (1 - \cos\varphi) \;+\; m_0 \cdot c^2} \;=\; f \cdot \frac{1}{1 \;+\; \frac{h \cdot f}{m_0 \cdot c^2} \cdot (1 - \cos\varphi)} \qquad (23.7)$$

$\frac{m_0 \cdot c^2}{h} \approx 1.236 \cdot 10^{20}$ Hz should then be called the *Compton frequency* of the electron.

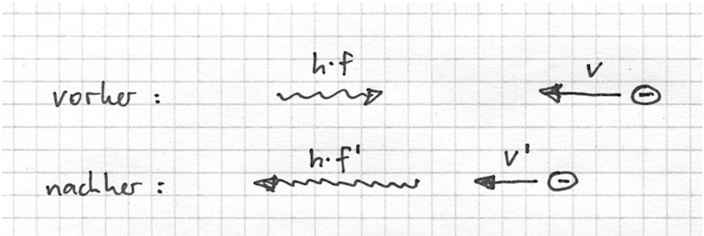
(23.4) and (23.6) both show that the energy of the scattered photon is always smaller than the energy of the infalling photon. Of course this is a basic consequence of conservation of energy. In addition, (23.6) shows

$$E' \;\geq\; E \cdot \frac{1}{1 \;+\; \frac{2 \cdot E}{m_0 \cdot c^2}}$$

An exact derivation of (23.4) without four-vectors is possible, but laborious. Compare e.g. https://www.physastromath.ch/uploads/myPdfs/Relativ/Relativ_03.pdf

B24 Inverse Compton Scattering

A photon can gain much energy by an elastic head-on collision with a fast electron :



The following four-vectors are used in the calculation :

- $P \;=\; \frac{h \cdot f}{c} \cdot (\,1\,,-1\,,0\,,0\,)$ for the photon before the collision
- $Q \;=\; \gamma_v \cdot m_0 \cdot (\,c\,,v\,,0\,,0\,)$ for the electron before the collision
- $R \;=\; \frac{h \cdot f'}{c} \cdot (\,1\,,1\,,0\,,0\,)$ for the photon after the collision
- $S \;=\; \gamma_{v'} \cdot m_0 \cdot (\,c\,,v'\,,0\,,0\,)$ for the electron after the collision

As usual we start with conservation of four-momentum : $P \,+\, Q \;=\; R \,+\, S$ (24.1)

squared $$P \circ P \,+\, 2 \cdot P \circ Q \,+\, Q \circ Q \;=\; R \circ R \,+\, 2 \cdot R \circ S \,+\, S \circ S$$

and evaluated $$0 \,+\, 2 \cdot P \circ Q \,+\, {m_0}^2 \cdot c^2 \;=\; 0 \,+\, 2 \cdot R \circ S \,+\, {m_0}^2 \cdot c^2$$

Hence we have $$P \circ Q \;=\; R \circ S \qquad (24.2)$$

Multiplying (24.1) by R $$P \circ R \,+\, Q \circ R \;=\; R \circ R \,+\, S \circ R \;=\; 0 \,+\, P \circ Q$$

we get $$P \circ R \,+\, Q \circ R \;=\; P \circ Q \qquad (24.3)$$

We have eliminated the unknown four-momentum S. Now we calculate the inner products :

- $P \circ R \;=\; \frac{h \cdot f}{c} \cdot \frac{h \cdot f'}{c} \cdot (1 + 1 - 0 - 0) \;=\; 2 \cdot \frac{h \cdot f}{c} \cdot \frac{h \cdot f'}{c}$
- $Q \circ R \;=\; R \circ Q \;=\; \frac{h \cdot f'}{c} \cdot \gamma_v \cdot m_0 \cdot (\,c - v\,)$
- $P \circ Q \;=\; \frac{h \cdot f}{c} \cdot \gamma_v \cdot m_0 \cdot (\,c + v\,)$

inserted in (24.3) $$2 \cdot \frac{h \cdot f}{c} \cdot \frac{h \cdot f'}{c} \,+\, \frac{h \cdot f'}{c} \cdot \gamma_v \cdot m_0 \cdot (\,c - v\,) \;=\; \frac{h \cdot f}{c} \cdot \gamma_v \cdot m_0 \cdot (\,c + v\,)$$

and divided by $\gamma_v \cdot m_0$ $$\frac{2}{\gamma_v \cdot m_0 \cdot c^2} \cdot h \cdot f \cdot h \cdot f' \,+\, h \cdot f' \cdot \left(1 - \frac{v}{c}\right) \;=\; h \cdot f \cdot \left(1 + \frac{v}{c}\right) \;\approx\; 2 \cdot h \cdot f \qquad (24.4)$$

For very fast electrons we have $v \approx c$. Then we may replace $\left(1 + \frac{v}{c}\right)$ by the number 2 to simplify the calculation. Dividing (24.4) by 2 we get

$$h \cdot f' \cdot \left(\frac{1}{\gamma_v \cdot m_0 \cdot c^2} \cdot h \cdot f \,+ \frac{1}{2} \cdot \left(1 - \frac{v}{c}\right) \right) = \; h \cdot f$$

$$h \cdot f' = \frac{h \cdot f}{\frac{1}{\gamma_v \cdot m_0 \cdot c^2} \cdot h \cdot f + \frac{1}{2} \cdot \left(1 - \frac{v}{c}\right)} = \gamma_v \cdot m_0 \cdot c^2 \cdot \frac{1}{1 + \frac{(c - v)}{2 \cdot c} \cdot \frac{\gamma_v \cdot m_0 \cdot c^2}{h \cdot f}} =$$

$$= \gamma_v \cdot m_0 \cdot c^2 \cdot \frac{1}{1 + \frac{(c - v) \cdot \gamma_v \cdot m_0 \cdot \lambda}{2 \cdot h}} \tag{24.5}$$

For $v \approx c$ we have in good *) approximation $\gamma_v \cdot (c - v) \approx c/(2 \cdot \gamma_v)$. This simplifies (24.5) to

$$E' = h \cdot f' \approx \gamma_v \cdot m_0 \cdot c^2 \cdot \frac{1}{1 + \frac{c \cdot m_0 \cdot \lambda}{4 \cdot \gamma_v \cdot h}} \tag{24.6}$$

Let us calculate an example value for $\gamma_v = 10'000$ and $\lambda = 500$ nm :

$$\frac{c \cdot m_0 \cdot \lambda}{4 \cdot \gamma_v \cdot h} \approx \frac{3 \cdot 10^8 \cdot 9.1 \cdot 10^{-31} \cdot 5 \cdot 10^{-7}}{8 \cdot 10^5 \cdot 6.6 \cdot 10^{-34}} \approx 0.517$$

and hence

$$E' \approx \frac{1}{1.517} \cdot 10'000 \cdot m_0 \cdot c^2 \approx 6'592 \cdot m_0 \cdot c^2 \approx 6'592 \cdot 511 \text{ keV} \approx 3.37 \text{ MeV}$$

With increasing energy of the pushing electron the fraction of energy that goes to the photon increases too. In that way quanta with very high energies are created.

γ_v	$E' / (m_0 \cdot c^2)$
10	0.019
100	5.27
1'000	162
10'000	6592
100'000	95'084

*)

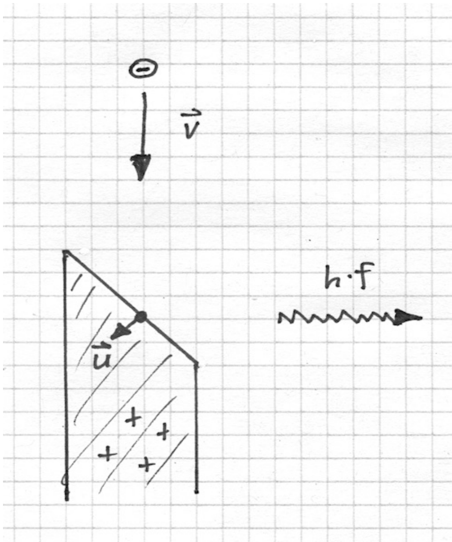
$$\gamma_v \cdot (c - v) = \frac{1}{\sqrt{1 - \frac{v}{c}}} \cdot \frac{1}{\sqrt{1 + \frac{v}{c}}} \cdot c \cdot \left(1 - \frac{v}{c}\right) = c \cdot \frac{\sqrt{1 - \frac{v}{c}}}{\sqrt{1 + \frac{v}{c}}} \approx c \cdot \frac{\sqrt{1 - \frac{v}{c}}}{\sqrt{2}} = \sqrt{c} \cdot \frac{\sqrt{c - v}}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{\sqrt{c}}{\sqrt{2} \cdot \gamma_v} \approx \frac{c - v}{\sqrt{c - v}} = \sqrt{c - v}$$

$$\Rightarrow \quad c - v \approx \frac{c}{2 \cdot {\gamma_v}^2} \quad \text{and} \quad \gamma_v \cdot (c - v) \approx \frac{c}{2 \cdot \gamma_v}$$

B25 Bremsstrahlung

An electron is accelerated in a vacuum tube by a tension of some ten kilovolts. When its flight ends on the metallic anode plate a great part of its kinetic energy is set free in form of an X-ray quantum. Before the collision we have a fast electron and a heavy atom at rest. After the collision we have the pushed atom, the electron and the high energy quantum. The atom and the electron go into the calculation as a single *cluster* :



So we have the four-vectors

- $P \;=\; \gamma_v \cdot m_0 \cdot (\, c \,, v \,, 0 \,, 0 \,)$ the electron before the collision
- $Q \;=\; 1 \cdot M \cdot (\, c \,, 0 \,, 0 \,, 0 \,)$ the atom before the collision
- $R \;=\; \gamma_u \cdot (M + m_0) \cdot (\, c \,, u_x \,, u_y \,, 0 \,)$ the atom and the electron after the collision
- $S \;=\; \frac{h \cdot f}{c} \cdot (\, 1 \,, 0 \,, 1 \,, 0 \,)$ the X-ray quantum

Once again the conservation of four-momentum :

$$P \,+\, Q \;=\; R \,+\, S \qquad (25.1)$$

And squared :

$$P \circ P \,+\, 2 \cdot P \circ Q \,+\, Q \circ Q \;=\; R \circ R \,+\, 2 \cdot R \circ S \,+\, S \circ S \qquad (25.2)$$

We have 6 inner products: $P \circ P = {m_0}^2 \cdot c^2$; $Q \circ Q = M^2 \cdot c^2$; $R \circ R = (M + m_0)^2 \cdot c^2$; $S \circ S = 0$;
$P \circ Q = \gamma_v \cdot m_0 \cdot M \cdot c^2$; $R \circ S = \gamma_u \cdot (M + m_0) \cdot \frac{h \cdot f}{c} \cdot (\, c - u_y)$.

Inserted in (25.2)

$${m_0}^2 \cdot c^2 + 2 \cdot \gamma_v \cdot m_0 \cdot M \cdot c^2 + M^2 \cdot c^2 = (M^2 + 2 \cdot M \cdot m_0 + {m_0}^2) \cdot c^2 + 2 \cdot \gamma_u \cdot (M + m_0) \cdot \frac{h \cdot f}{c} \cdot (\, c - u_y)$$

simplified
$$2 \cdot \gamma_v \cdot m_0 \cdot M \cdot c^2 \;=\; 2 \cdot M \cdot m_0 \cdot c^2 + 2 \cdot \gamma_u \cdot (M + m_0) \cdot h \cdot f \cdot \left(1 - \frac{u_y}{c}\right)$$

again
$$(\gamma_v - 1) \cdot m_0 \cdot M \cdot c^2 \;=\; \gamma_u \cdot (M + m_0) \cdot h \cdot f \cdot \left(1 - \frac{u_y}{c}\right)$$

and finally
$$(\gamma_v - 1) \cdot m_0 \cdot c^2 \cdot \frac{M}{\gamma_u \cdot (M + m_0)} \cdot \frac{c}{c - u_y} \;=\; h \cdot f \qquad (25.3)$$

If the anode is made from molybdenum e.g. we have $\frac{M}{M + m_0} \approx \frac{175000}{175001}$. The factor $\frac{M}{M + m_0}$ comes very close to 1. Further we have $u_y < u \ll c$, γ_u and the factor $c/(c - u_y)$ are only a little bit greater than 1. The less energy is absorbed by the atom the smaller is the difference of those factors to 1 . So, as an upper limit for the energy of the X-ray quantum, we find

$$h \cdot f \;\leq\; (\gamma_v - 1) \cdot m_0 \cdot c^2 \;=\; E_{kin}$$

The result is far from being a surprise: The maximum energy of the quantum is 100% of the kinetic energy of the electron. We might have predicted that result right from the beginning :

$$h \cdot f_{max} \;=\; E_{kin} \;=\; U \cdot e$$

Crystal structures are analyzed by diffractometers at tensions of 20 to 40 kilovolts, corresponding to X-ray wavelengths of one or a half Angström.

C26 Lorentz Force as a Four-Vector

Maxwell's theory of electromagnetism is perfectly compatible with STR. So we may expect the Lorentz force law to be valid in STR, too: $\vec{f} = q \cdot (\vec{E} + \vec{u} \times \vec{B})$. Basically, this force law gives the definition of the electric and the magnetic field vectors. Neglecting gravity, the total force acting on a charged particle consists of Coulomb force and Lorentz force.

(6.3) tells us how to define the corresponding four-force:

$$K = \gamma \cdot (\frac{1}{c} \cdot \frac{dE_{tot}}{dt}, \ \frac{d\vec{p}}{dt}) = \gamma \cdot (\frac{1}{c} \cdot \vec{f} \cdot \vec{u}, \ \vec{f}) = \gamma \cdot q \cdot (\frac{1}{c} \cdot \vec{E} \cdot \vec{u}, \ \vec{E} + \vec{u} \times \vec{B}) \qquad (26.1)$$

The magnetic field is not involved in changing the particles energy: $\vec{u} \times \vec{B}$ and $\vec{u}$ are always perpendicular to each other, and so $\vec{f} \cdot \vec{u} = q \cdot (\vec{E} + \vec{u} \times \vec{B}) \cdot \vec{u} = q \cdot \vec{E} \cdot \vec{u}$.

K can be written as the product of a matrix F and the four-velocity U :

$$K = \frac{q}{c} \cdot \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & c \cdot B_z & -c \cdot B_y \\ E_y & -c \cdot B_z & 0 & c \cdot B_x \\ E_z & c \cdot B_y & -c \cdot B_x & 0 \end{pmatrix} \cdot \gamma \cdot \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix} \qquad (26.2)$$

The matrix is named F in honor of Michael Faraday. F is the SRT standard description of the electromagnetic field. Using the symbol F we can rewrite the Lorentz force law as

$$K = \frac{q}{c} \cdot F \cdot U \qquad (26.3)$$

K and U are four-vectors. In another frame S' they are given by $K' = L \cdot K$ and $U' = L \cdot U$. It is easy to find the matrix F' with

$$K' = \frac{q}{c} \cdot F' \cdot U'$$

We multiply (26.3) from the left side with our matrix L from **A1** and we get

$$L \cdot K = \frac{q}{c} \cdot L \cdot F \cdot U = \frac{q}{c} \cdot L \cdot F \cdot L^{-1} \cdot L \cdot U$$

and hence

$$K' = L \cdot K = \frac{q}{c} \cdot (L \cdot F \cdot L^{-1}) \cdot (L \cdot U) = \frac{q}{c} \cdot F' \cdot U'$$

The matrix F' is given by $F' = L \cdot F \cdot L^{-1}$. (26.4)

If F is the description of some electromagnetic field in sytem S then $F' = L \cdot F \cdot L^{-1}$ is the description of the same electromagnetic field in a system S', that moves with speed v relative to S in positive x-direction. In the next section we will calculate the corresponding transformations of the single components of the electric and the magnetic field vectors.

C27 The Transformation of the Electromagnetic Field Vectors

Following equation (26.4) we just have to calculate $L \cdot F \cdot L^{-1}$ to get the components of F' :

With $F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & c \cdot B_z & -c \cdot B_y \\ E_y & -c \cdot B_z & 0 & c \cdot B_x \\ E_z & c \cdot B_y & -c \cdot B_x & 0 \end{pmatrix}$ and $L = \begin{pmatrix} \gamma & -\gamma \cdot \beta & 0 & 0 \\ -\gamma \cdot \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

we find

$$F' = \begin{pmatrix} 0 & E_x & \gamma_v \cdot (E_y - v \cdot B_z) & \gamma_v \cdot (E_z + v \cdot B_y) \\ E_x & 0 & c \cdot \gamma_v \cdot (B_z - \frac{v}{c^2} \cdot E_y) & -c \cdot \gamma_v \cdot (B_y - \frac{v}{c^2} \cdot E_z) \\ \gamma_v \cdot (E_y - v \cdot B_z) & -c \cdot \gamma_v \cdot (B_z - \frac{v}{c^2} \cdot E_y) & 0 & c \cdot B_x \\ \gamma_v \cdot (E_z + v \cdot B_y) & c \cdot \gamma_v \cdot (B_y - \frac{v}{c^2} \cdot E_z) & -c \cdot B_x & 0 \end{pmatrix}$$

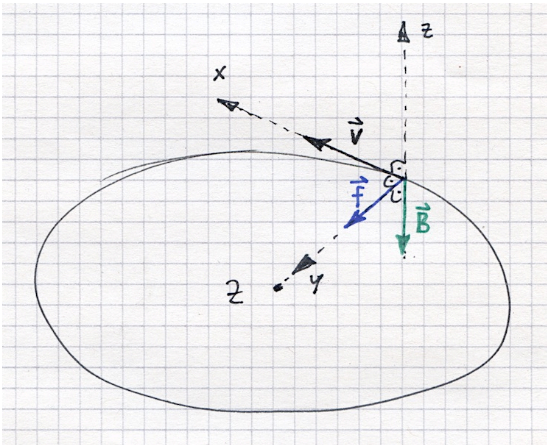
Hence we have

$$\begin{aligned} E_x{}' &= E_x & B_x{}' &= B_x \\ E_y{}' &= \gamma_v \cdot (E_y - v \cdot B_z) & B_y' &= \gamma_v \cdot \left(B_y + \frac{v}{c^2} \cdot E_z\right) \\ E_z{}' &= \gamma_v \cdot (E_z + v \cdot B_y) & B_z' &= \gamma_v \cdot \left(B_z - \frac{v}{c^2} \cdot E_y\right) \end{aligned} \tag{27.1}$$

To find the reverse transformation we have to replace v by $-v$ and, by that, L by L^{-1} . Doing so the plus and minus signs in the second and third row of (27.1) are exchanged.

In STR, the electric and the magnetic field are united to a single electromagnetic field. A pure eletric field in system S shows up as a mixed electric and magnetic field in system S'. With that, Einstein got rid of the 'asymmetries' he is complaining about in the first sentence of his famous 1905 paper "Zur Elektrodynamik bewegter Körper": "Dass die Elektrodynamik Maxwells - wie dieselbe gegenwärtig aufgefasst zu werden pflegt - zu Asymmetrien führt, welche den Phänomenen nicht anzuhaften scheinen, ist bekannt."

C28 Force and Acceleration in a Storage Ring



In the laboratory frame S a particle with positive charge q is caught in a storage ring. The only non-zero component of the electromagnetic field is $B_z = -B$. The Lorentz force keeps the particle on its circular trajectory :

$$\vec{f} = q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) = q \cdot (\, 0 \,, v \cdot B , 0\,)$$

Force and acceleration are perpendicular to the speed $\vec{v}$, so we can use (7.7) and write

$$\vec{f} = \gamma_v \cdot m_0 \cdot \vec{a}$$

hence $\quad \vec{a} = (\, 0 \,, a_y \,, 0\,) \quad$ with $\quad a_y = \frac{q \cdot v \cdot B}{\gamma_v \cdot m_0} \qquad$ (28.1)

With $a_y = \frac{v^2}{r}$ we further find $\quad B = \frac{\gamma_v \cdot v \cdot m_0}{q \cdot r} = \frac{p}{q \cdot r} \qquad$ (28.2)

In the CERN laboratories near Geneva protons are accelerated up to 299'780'455 m/s (the speed of light is 299'792'458 m/s). They are kept in a storage ring with a diameter of 4243 m. Thus, $\gamma_v \approx 111.75$, and the strength of the field B needed to keep them on track is by a factor 112 greater than non-STR calculation would suggest. Instead of some milli-Teslas the field strength created by superconducting magnets goes up to 8.3 Tesla.

We recalculate a_y in the laboratory system using the formalism of four-vectors. With (7.5) we have

$$K = \frac{q}{c} \cdot F \cdot U = \gamma_v \cdot \frac{q}{c} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -c \cdot B & 0 \\ 0 & c \cdot B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c \\ v \\ 0 \\ 0 \end{pmatrix} = \gamma_v \cdot \frac{q}{c} \cdot \begin{pmatrix} 0 \\ 0 \\ v \cdot c \cdot B \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma_v \cdot v \cdot q \cdot B \\ 0 \end{pmatrix} = m_0 \cdot {\gamma_v}^2 \cdot \begin{pmatrix} 0 \\ a_x \\ a_y \\ a_z \end{pmatrix}$$

and we again find the result (28.1) .

A third calculation is done in the system S' of the moving particle, in its actual comoving inertial frame. There, the particles four-velocity is $U' = (c, \vec{u}) = (c, 0,0,0)$ and, following (10.8), its proper acceleration is $A' = {\gamma_u}^2 \cdot (0, a_x, a_y, a_z)$. We get the same result by the matrix multiplication $A' = L \cdot A$:

$$A' = \begin{pmatrix} \gamma & -\gamma \cdot \beta & 0 & 0 \\ -\gamma \cdot \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ {\gamma_v}^2 \cdot a_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ {\gamma_v}^2 \cdot a_y \\ 0 \end{pmatrix} = {\gamma_u}^2 \cdot \begin{pmatrix} 0 \\ a_x{}' \\ a_y{}' \\ a_z{}' \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 \\ a_x{}' \\ a_y{}' \\ a_z{}' \end{pmatrix}$$

So we find $\quad A' = A \quad$ and $\quad \vec{a}' = {\gamma_v}^2 \cdot \vec{a} = (\, 0 \,, \frac{\gamma_v \cdot q \cdot v \cdot B}{m_0} \,, 0\,) \qquad$ (28.3)

$K' = m_0 \cdot A' = m_0 \cdot A = K \quad$ and (7.7) induce $\quad \vec{f}' = m_0 \cdot \gamma_u \cdot \vec{a}' = m_0 \cdot 1 \cdot \vec{a}' = m_0 \cdot {\gamma_v}^2 \cdot \vec{a} = \gamma_v \cdot \vec{f} \qquad$ (28.4)

We do the calculation one more time in system S', but now we will use the formulas (27.1) for the transformations of the electromagnetic field :

$$E_x{}' = E_x = 0 \qquad\qquad B_x{}' = B_x = 0$$

$$E_y{}' = \gamma_v \cdot (E_y - v \cdot B_z) = \gamma_v \cdot v \cdot B \qquad\qquad B_y' = \gamma_v \cdot \left(B_y + \frac{v}{c^2} \cdot E_z\right) = 0$$

$$E_z{}' = \gamma_v \cdot (E_z + v \cdot B_y) = 0 \qquad\qquad B_z' = \gamma_v \cdot \left(B_z - \frac{v}{c^2} \cdot E_y\right) = -\gamma_v \cdot B$$

We find

$$\vec{f}' = q \cdot (\vec{E}' + \vec{u} \times \vec{B}') = q \cdot (\vec{E}' + \vec{0} \times \vec{B}') = q \cdot (0, \gamma_v \cdot v \cdot B, 0) = m_0 \cdot \gamma_u \cdot \vec{a}' = m_0 \cdot 1 \cdot \vec{a}'$$

and finally again $$\vec{a}' = \frac{\gamma_v \cdot v \cdot B}{m_0} = \gamma_v{}^2 \cdot \vec{a} \qquad (28.3) = (28.5)$$

If a central force is at work we always have $\vec{f}' = m_0 \cdot \vec{a}' = m_0 \cdot \gamma_v{}^2 \cdot \vec{a} = \gamma_v \cdot (\gamma_v \cdot m_0 \cdot \vec{a}) = \gamma_v \cdot \vec{f}$, and in the eigen system of the moving particle we have

$$\vec{a}' = \gamma_v{}^2 \cdot \vec{a} \qquad \text{and} \qquad \vec{f}' = \gamma_v \cdot \vec{f} \qquad (28.6)$$

The general explanation is given with

$$f_y{}' = \frac{d}{dt'}(p_y{}') = \frac{d}{d\tau}(p_y) = \frac{d}{dt}(p_y) \cdot \frac{dt}{d\tau} = \gamma_v \cdot \frac{d}{dt}(p_y) = \gamma_v \cdot f_y$$

Similarly we have

$$a_y{}' = \frac{d}{d\tau}\left(\frac{dy'}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{dy}{dt} \cdot \frac{dt}{d\tau}\right) = \frac{d}{d\tau}\left(\gamma_v \cdot \frac{dy}{dt}\right) = \gamma_v \cdot \frac{d}{d\tau}\left(\frac{dy}{dt}\right) = \gamma_v \cdot \frac{d}{dt}\left(\frac{dy}{dt}\right) \cdot \frac{dt}{d\tau} = \gamma_v{}^2 \cdot \frac{d}{dt}\left(\frac{dy}{dt}\right) = \gamma_v{}^2 \cdot a_y$$

If the force is perpendicular to velocity, γ_v is a constant term and thus does not influence the differentiation with respect to time.

So much for the special case of forces perpendicular to velocity. The other special case with $\vec{f}$ parallel to $\vec{v}$ is treated in the next section.

C29 Force and Acceleration in a Linear Accelerator

A particle with rest mass m_0 and charge q gets accelerated along the x-direction in laboratory system S by a constant electric field $\vec{E} = E_x$. There is no magnetic field at work in frame S .

Following (7.6) we have

$$\vec{f} = q \cdot \vec{E} = q \cdot \vec{E}_x = \vec{f}_x = m_0 \cdot \gamma^3 \cdot \vec{a}_x$$

and hence

$$\frac{q \cdot E_x}{m_0} = \gamma^3 \cdot a_x = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} \cdot \frac{dv}{dt}$$

resulting in

$$\frac{q \cdot E_x}{m_0} \cdot dt = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} \cdot dv \qquad (29.1)$$

By integration (Bronstein integral Nr. 178) we get

$$\frac{q \cdot E_x}{m_0} \cdot t + C = v \cdot \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \qquad (29.2)$$

The constant term C disappears if the speed is zero at time $t = 0$. Solving (29.2) for v with $C = 0$ we get

$$v(t) = \frac{\frac{q \cdot E_x}{m_0} \cdot t}{\sqrt{1 + \left(\frac{q \cdot E_x}{m_0 \cdot c}\right)^2 \cdot t^2}} \qquad (29.3)$$

In the beginning we have a linear growth in velocity as we would have in classical physics. Then, the denominator slows down the increase of speed more and more, and in the limit of $t \to \infty$ the speed approaches the speed of light c :

$$\lim_{t \to \infty} \frac{c \cdot \frac{q \cdot E_x}{m_0}}{\sqrt{\frac{c^2}{t^2} + \left(\frac{q \cdot E_x}{m_0}\right)^2}} = \frac{c \cdot \frac{q \cdot E_x}{m_0}}{\sqrt{\left(\frac{q \cdot E_x}{m_0}\right)^2}} = c \qquad (29.4)$$

To get (29.4) we did multiply nominator and denominator of (29.3) by c/t .

What would be the description of this process in the comoving frame S' of the accelerated particle ? In each moment we have in S' $E_x{}' = E_x$, $E_y{}' = 0$, $E_z{}' = 0$, $B_x{}' = B_x = 0$, $B_y{}' = 0$ and $B_z{}' = 0$. In any instant we have the very same situation as in system S , the equations (29.1) and (29.2) are valid in S' .

Let us study the same situation again in the laboratory frame S , but now using four-vectors.

We have $\vec{E} = (E_x, 0, 0)$, $\vec{B} = 0$ and $\vec{v} = \vec{v}(t) = (v(t), 0, 0)$. The vectors $\vec{f}$, $\vec{v}$ and $\vec{a}$ are parallel to each other. Further we have $\vec{f} = q \cdot (\vec{E} + \vec{v} \times \vec{B}) = q \cdot (E_x, 0, 0)$ and $\vec{f} \cdot \vec{v} = q \cdot v \cdot E_x$. Inserting all that in equation (7.5) we get

$$K = \gamma \cdot \begin{pmatrix} \frac{1}{c} \cdot \vec{f} \cdot \vec{v} \\ f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} \gamma \cdot \frac{1}{c} \cdot q \cdot v \cdot E_x \\ \gamma \cdot q \cdot E_x \\ 0 \\ 0 \end{pmatrix} = m_0 \cdot A = m_0 \cdot \begin{pmatrix} \gamma^4 \cdot \frac{1}{c} \cdot v \cdot a_x \\ \gamma^4 \cdot \frac{1}{c^2} \cdot v^2 \cdot a_x + \gamma^2 \cdot a_x \\ 0 \\ 0 \end{pmatrix} \tag{29.5}$$

From (29.5) we get two equations: One for the temporal components and one for the first spatial components. The temporal equation, a little bit simplified, is

$$q \cdot E_x = m_0 \cdot \gamma^3 \cdot a_x$$

and we are back to (29.1) and (7.6). The spatial equation is

$$f_x = q \cdot E_x = m_0 \cdot \gamma^3 \cdot \frac{1}{c^2} \cdot v^2 \cdot a_x + \gamma \cdot a_x = m_0 \cdot \gamma^3 \cdot a_x \cdot \left(\frac{v^2}{c^2} + \gamma^{-2}\right)$$

The left sides of both equations are identical, so the factor $\left(\frac{v^2}{c^2} + \gamma^{-2}\right)$ has to be equal to 1 :

$$\frac{v^2}{c^2} + \gamma^{-2} = \frac{v^2}{c^2} + \left(1 - \frac{v^2}{c^2}\right) = 1$$

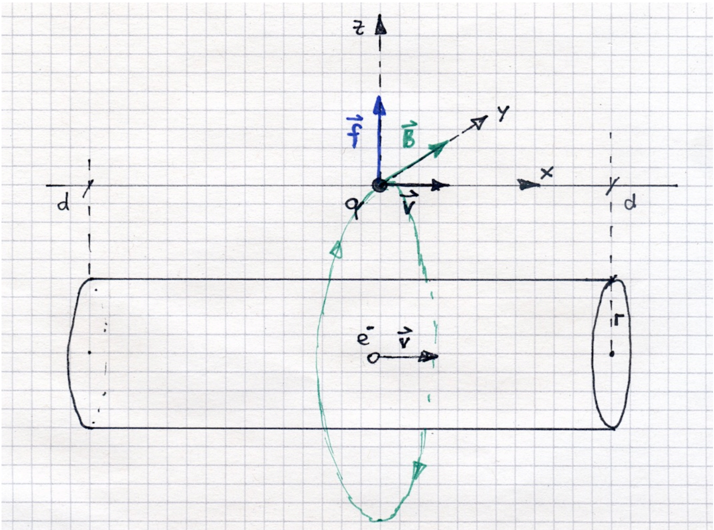
We got the same equation twice for a_x .

C30 The Cylindrical Conductor 1

Let a current I flow in a long cylindrical wire. The wire is at rest in the laboratory frame S, it's cross section has radius r and the mean drift velocity of the electrons is v . If n stands for the number of electrons per unit volume in the conduction band our variables are connected by

$$I = n \cdot e \cdot r^2 \cdot \pi \cdot v \tag{30.1}$$

Outside the wire, there is no force acting on a charged particle at rest, the electrical field is zero. However, we have a magnetic field. Its lines of force are concentric circles as depicted in green in the following figure :



The current flows from the right to the left, the electrons are drifting with speed v from left to the right. The symmetry of the magnetic field reflects the symmetry of the conductor and the current.

For the strength of the magnetic field in distance d to the center of the wire we find with Ampère's law and (30.1)

$$B_y = \frac{\mu_0}{2\cdot\pi} \cdot \frac{I}{d} = \frac{\mu_0}{2\cdot\pi} \cdot \frac{1}{d} \cdot n \cdot e \cdot r^2 \cdot \pi \cdot v \tag{30.2}$$

For a charged particle in distance d to the center of the wire, that moves with the same speed v as the electrons in the x-direction we have

$$\vec{f} = q \cdot (\vec{v} \times \vec{B}) = f_z = q \cdot v_x \cdot B_y = q \cdot v \cdot \frac{\mu_0}{2\cdot\pi} \cdot \frac{1}{d} \cdot n \cdot e \cdot r^2 \cdot \pi \cdot v = \frac{\mu_0}{2} \cdot \frac{1}{d} \cdot q \cdot n \cdot e \cdot r^2 \cdot v^2 \tag{30.3}$$

Now let us switch to the proper system S' of the moving particle. In S' the speed of the particle is zero, hence no Lorentz force can be at work. If the particle gets accelerated in z-direction the wire has to carry an overall positive charge producing an electric field. Indeed, we will find this to be true, and the reason lies in Lorentz contraction: Amazingly, the small drift velocitiy in the order of magnitude of 1 mm per second produces a macroscopic effect via Lorentz contraction!

In system S total charge of the wire is zero. There is no electric field outside the wire. Thus, the driftig electrons necessarily have the same charge density as the positively charged atoms at rest : $\rho_+ + \rho_- = n \cdot e + n \cdot (-e) = 0$. In system S' the distance of the drifting electrons is no longer Lorentz contracted, but the distance of the atoms sitting in their lattice is Lorentz contracted now. Hence we have for the charge densities in S'

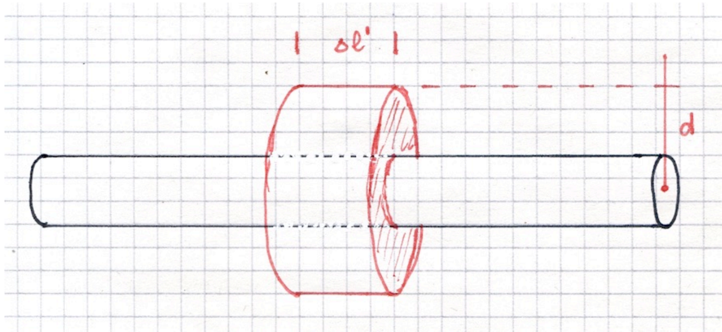
$$\rho' = \rho_+' + \rho_-' = n \cdot e \cdot \gamma_v + n \cdot (-e) \cdot \frac{1}{\gamma_v} = n \cdot e \cdot \left(\gamma_v - \frac{1}{\gamma_v}\right) = n \cdot e \cdot \gamma_v \cdot {\beta_v}^2 \tag{30.4}$$

It is this surplus positive charge that induces the electric field and the force on our charged particle.

The strength of the electric field in distance $d = d'$ from the center of the wire is calculated with the law of Gauss :

$$\oint \vec{E}' \cdot d\vec{A'} = \frac{1}{\varepsilon_0} \cdot \int \rho' \cdot dV'$$

On the left, the integral runs over the *surface* of some closed space; on the right the integral goes over the *volume* of that space. For that closed space we choose a cylinder of radius d and arbitrary length $\Delta l'$ sharing its symmetry axis with the wire :



For reasons of symmetry the electric field has to be zero in the x-direction, the charge density shows the same symmetry as the wire. Before, in system S, the positive x-direction was distinguished by the current.
Hence, the circular areas of the red cylinder do not contribute to the left integral. On the lateral surface, $\vec{E}'$ is vertical to the surface, and its absolute value is the same everywhere. Thus, the left integral yields

$$\oint \vec{E}' \cdot d\vec{A} = E' \cdot 2 \cdot \pi \cdot d \cdot \Delta l'$$

The charge density outside the wire is zero. So, for the integral on the right side, we just have to integrate over the volume of the wire enclosed in the cylinder. The charge density is given by (30.4), and we get

$$\frac{1}{\varepsilon_0} \cdot \int \rho' \cdot dV' = \frac{1}{\varepsilon_0} \cdot \rho' \cdot r^2 \cdot \pi \cdot \Delta l' = \frac{1}{\varepsilon_0} \cdot r^2 \cdot \pi \cdot \Delta l' \cdot n \cdot e \cdot \gamma_v \cdot {\beta_v}^2$$

So we get from the law of Gauss $\quad E' \cdot 2 \cdot \pi \cdot d \cdot \Delta l' = \frac{1}{\varepsilon_0} \cdot r^2 \cdot \pi \cdot \Delta l' \cdot n \cdot e \cdot \gamma_v \cdot {\beta_v}^2$

Solved for E'

$$E' = \frac{1}{\varepsilon_0} \cdot \frac{1}{2 \cdot d} r^2 \cdot n \cdot e \cdot \gamma_v \cdot {\beta_v}^2 \tag{30.5}$$

In system S' we have a Coulomb force $\vec{f}'$ acting on our charged particle with

$$\vec{f}' = q \cdot \vec{E}' = f_z' = q \cdot E' = q \cdot \frac{1}{\varepsilon_0} \cdot \frac{1}{2 \cdot d} r^2 \cdot n \cdot e \cdot \gamma_v \cdot {\beta_v}^2$$

With $c^2 = 1/(\varepsilon_0 \cdot \mu_0)$ and ${\beta_v}^2 = v^2/c^2$ we get

$$f' = q \cdot E' = q \cdot \mu_0 \cdot \frac{1}{2 \cdot d} r^2 \cdot n \cdot e \cdot \gamma_v \cdot v^2 \tag{30.6}$$

Comparing with (30.3) we find $f' = \gamma_v \cdot f$. The factor γ_v shows up here for the same reason as in (28.6) .

This section follows quite closely the presentation in [2 - 5].

C31 The Cylindrical Conductor 2

We consider the same situation as in the prevous section (first figure of **C30**). A particle with positive charge q moves parallel to a long conducting wire with the same velocity as the drifting electrons within that wire.

In **C30** we noted that in the laboratory frame S of the wire a Lorentz force acts on the particle pushing it away from the wire. Following (30.3) we have

$$\vec{f} = f_z = q \cdot v \cdot B_y = q \cdot v \cdot \frac{\mu_0}{2 \cdot \pi} \cdot \frac{I}{d} \tag{31.1}$$

Once again we calculate the force $\vec{f}'$ in the proper system S' of that particle. This time we use the transformation rules of the electromagnetic field.

In system S we have $\vec{E} = 0$ and, at the position of the particle, $\vec{B} = B_y = \frac{\mu_0}{2 \cdot \pi} \cdot \frac{I}{d}$. From that we get with (27.1)

$$E_x{}' = E_x = 0 \qquad B_x{}' = B_x = 0$$

$$E_y{}' = \gamma_v \cdot (E_y - v \cdot B_z) = 0 \qquad B_y{}' = \gamma_v \cdot \left(B_y + \frac{v}{c^2} \cdot E_z\right) = \gamma_v \cdot B_y$$

$$E_z{}' = \gamma_v \cdot (E_z + v \cdot B_y) = \gamma_v \cdot v \cdot B_y \qquad B_z{}' = \gamma_v \cdot \left(B_z - \frac{v}{c^2} \cdot E_y\right) = 0$$

The force acting in S' on the particle is

$$\vec{f}' = q \cdot (\vec{E}' + \vec{u} \times \vec{B}') = q \cdot (\vec{E}' + \vec{0} \times \vec{B}') = q \cdot E_z{}'$$

For the Coulomb force $\vec{f}'$ we find

$$\vec{f}' = f_z{}' = \gamma_v \cdot q \cdot v \cdot B_y = \gamma_v \cdot \vec{f} \tag{31.2}$$

Virtually without any effort we could verify the result of the last section.

So we have $f' = \gamma_v \cdot q \cdot v \cdot \frac{\mu_0}{2 \cdot \pi} \cdot \frac{I}{d}$ and $f = q \cdot v \cdot \frac{\mu_0}{2 \cdot \pi} \cdot \frac{I}{d}$ where I stands for the electric current measured in system S .

What about the current measured in system S' ? Electric current is the amout of electric charge passing a cross section of the conductor per time unit. The calculation is easy for directions perpendicular to the relative speed v :

$$I_y{}' = dQ'/dt' = \rho' \cdot A' \cdot u_y{}' = \gamma_v \cdot \rho \cdot \frac{1}{\gamma_v} \cdot A \cdot \gamma_v \cdot u_y = \gamma_v \cdot \rho \cdot A \cdot u_y = \gamma_v \cdot I_y \tag{31.3}$$

u_y denotes the drift velocity of the electrons in y-direction. In general, the transformation of that drift velocity in x-direction is a bit more complicated. However, in our special case, the drift velocity $u_x{}'$ of the charge in S' is well known: $u_x{}' = -v$. In this special case we find using (30.4) and (30.1)

$$I_x{}' = \rho' \cdot A' \cdot u_x{}' = \rho' \cdot A' \cdot (-v) = \gamma_v \cdot n \cdot e \cdot \gamma_v \cdot \beta_v{}^2 \cdot A \cdot (-v) = \gamma_v \cdot (-\rho) \cdot A \cdot (-v) = \gamma_v \cdot I_x \tag{31.4}$$

and we have

$$f' = q \cdot v \cdot \frac{\mu_0}{2 \cdot \pi} \cdot \frac{I'}{d'} \quad \text{and} \quad f = q \cdot v \cdot \frac{\mu_0}{2 \cdot \pi} \cdot \frac{I}{d} \quad \text{with} \quad d' = d \ , \ I' = \gamma_v \cdot I \quad \text{and} \quad B_y{}' = \gamma_v \cdot B_y \tag{31.5}$$

This result is restricted to our special case of $u^2 = v^2$.

C32 The Cylindrical Conductor 3

One more time we are back in the situation of **C30**. This time we will calculate the force acting on our particle in system S' using the four-currents J and J' . The point is to illustrate the concept of current density.

The overal charge of the conducting wire is zero. There is no electric field outside the wire. Thus the charge density of the drifting electrons has to be the same as the charge density of the lattice atoms deprived of their electrons in the conduction band. Let us denote these charge densities with ρ and $-\rho$. Total four-current J in system S is the sum of the four-current J_- of the electrons in the conduction band and the four-current J_+ of the lattice atoms :

$$J_{tot} = \begin{pmatrix} \rho_{tot} \cdot c \\ j_x \\ j_y \\ j_z \end{pmatrix} = J_+ + J_- = \rho \cdot \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} + (-\rho) \cdot \begin{pmatrix} c \\ v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho \cdot v \\ 0 \\ 0 \end{pmatrix}$$

The drift velocity u of the electrons is by assumption the same as the velocity v of system S' as seen from S.
The electric current associated with J_{tot} is

$$I = I_x = j_x \cdot A_x = -\rho \cdot v \cdot r^2 \cdot \pi \tag{32.1}$$

We have

$$J_+ = \rho \cdot \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} = \rho_{0+} \cdot 1 \cdot \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} = \rho_{0+} \cdot \gamma_u \cdot \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix} = \rho_{0+} \cdot U_+ \qquad \text{with} \qquad J_+ \circ J_+ = {\rho_{0+}}^2 \cdot c^2$$

and

$$J_- = -\rho \cdot \begin{pmatrix} c \\ v \\ 0 \\ 0 \end{pmatrix} = \rho_{0-} \cdot \gamma_v \cdot \begin{pmatrix} c \\ v \\ 0 \\ 0 \end{pmatrix} = \rho_{0-} \cdot \gamma_v \cdot \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix} = -\rho_{0-} \cdot U_- \qquad \text{with} \qquad J_- \circ J_- = {\rho_{0-}}^2 \cdot c^2$$

ρ_{0+} and ρ_{0-} are the charge densities as measured in their proper system.

Now we calculate the four-currents J', ${J_+}'$ and ${J_-}'$ in system S' by multiplying the four-currents in system S with the Lorentz matrix L :

$${J_+}' = L \cdot J_+ = \begin{pmatrix} \gamma & -\gamma \cdot \beta & 0 & 0 \\ -\gamma \cdot \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \rho \cdot \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} = \rho \cdot \gamma \cdot \begin{pmatrix} c \\ -v \\ 0 \\ 0 \end{pmatrix}$$

$${J_-}' = L \cdot J_- = \begin{pmatrix} \gamma & -\gamma \cdot \beta & 0 & 0 \\ -\gamma \cdot \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot (-\rho) \cdot \begin{pmatrix} c \\ v \\ 0 \\ 0 \end{pmatrix} = -\rho \cdot \gamma \cdot \begin{pmatrix} c - \beta \cdot v \\ -\beta \cdot c + v \\ 0 \\ 0 \end{pmatrix} = -\rho \cdot \gamma \cdot \begin{pmatrix} c - \beta \cdot v \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and hence

$$J' = {J_+}' + {J_-}' = \rho \cdot \gamma \cdot \begin{pmatrix} c \\ -v \\ 0 \\ 0 \end{pmatrix} - \rho \cdot \gamma \cdot \begin{pmatrix} c - \beta \cdot v \\ 0 \\ 0 \\ 0 \end{pmatrix} = \rho \cdot \gamma \cdot \begin{pmatrix} \beta \cdot v \\ -v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c \cdot \rho'_{tot} \\ {j_x}' \\ {j_y}' \\ {j_z}' \end{pmatrix}$$

In S' the current density of the electrons is zero, but the positively charged lattice atoms represent an electric current in the negative x'-direction. The strength of that current is

$$I' = {I_x}' = {j_x}' \cdot {A_x}' = \rho \cdot \gamma \cdot (-v) \cdot r^2 \cdot \pi = \gamma \cdot I \tag{32.2}$$

Following the law of Ampère, the current I' induces in distance $d' = d$ a magnetic field of strenth

$$B_y{}' = \frac{\mu_0}{2 \cdot \pi} \cdot \frac{I'}{d'} = \frac{\mu_0}{2 \cdot \pi} \cdot \frac{\gamma \cdot I}{d} = \gamma \cdot \frac{\mu_0}{2 \cdot \pi} \cdot \frac{I}{d} = \gamma \cdot B_y \tag{32.3}$$

but that magnetic field does not act on our charged particle at rest in system S' !

In system S', total charge density of the wire is

$$\rho'_{tot} = \rho \cdot \gamma \cdot \beta \cdot v \cdot \tfrac{1}{c} = \rho \cdot \gamma \cdot \beta^2 \tag{32.4}$$

As demonstrated in **C30** this charge density exerts a Coulomb force on our charged particle of amount

$$E' \cdot 2 \cdot \pi \cdot d' \cdot \Delta l' = \frac{1}{\varepsilon_0} \cdot \rho'_{tot} \cdot r^2 \cdot \pi \cdot \Delta l'$$

Rearranging the terms we find

$$E' = \frac{1}{2 \cdot \pi \cdot \varepsilon_0} \cdot \frac{1}{d'} \cdot \rho'_{tot} \cdot r^2 \cdot \pi = \frac{1}{2 \cdot \pi \cdot \varepsilon_0} \cdot \frac{1}{d'} \cdot \rho \cdot \gamma \cdot \beta^2 \cdot r^2 \cdot \pi = -\frac{1}{2 \cdot \pi \cdot \varepsilon_0} \cdot \frac{1}{d'} \cdot \beta^2 \cdot \frac{1}{v} \cdot I' =$$

$$= -\frac{1}{2 \cdot \pi \cdot \varepsilon_0} \cdot \frac{1}{d'} \cdot \frac{v^2}{c^2} \cdot \frac{1}{v} \cdot I' = -\frac{\mu_0 \cdot \varepsilon_0}{2 \cdot \pi \cdot \varepsilon_0} \cdot \frac{1}{d'} \cdot v \cdot I' = -\frac{\mu_0}{2 \cdot \pi} \cdot \frac{I'}{d'} \cdot v = B_y{}' \cdot v = \gamma \cdot B_y \cdot v \tag{32.5}$$

For the Coulomb force in system S' we find as before in (31.2)

$$\vec{f}' = f_z{}' = q \cdot E' = q \cdot \gamma_v \cdot q \cdot v \cdot B_y = \gamma_v \cdot \vec{f} \tag{32.6}$$

The four-current $J = (0, -\rho \cdot v, 0, 0)^T$ is given from the beginning. Hence $J' = L \cdot J$ is straightway calculated and we find the current density and the charge density in system S' on a short path by

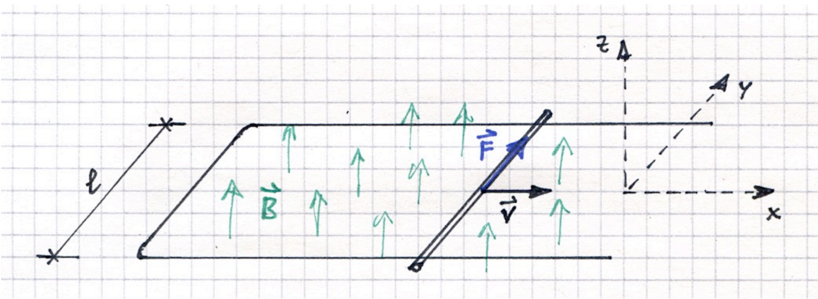
$$J' = L \cdot J = \begin{pmatrix} \gamma & -\gamma \cdot \beta & 0 & 0 \\ -\gamma \cdot \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -\rho \cdot v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \rho \cdot v \cdot \gamma \cdot \beta \\ -\rho \cdot \gamma \cdot v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c \cdot \rho'_{tot} \\ j_x{}' \\ j_y{}' \\ j_z{}' \end{pmatrix} = \begin{pmatrix} c \cdot (\rho \cdot \gamma \cdot \beta^2) \\ -\rho \cdot \gamma \cdot v \\ 0 \\ 0 \end{pmatrix}$$

As shown above we get from that the field vectors B' and E' of the conductor in system S' . All results are in complete agreement with those calculated in sections **C28** and **C29** .

We made some effort to show that four-currents in general cannot be written in the form $J = \rho_0 \cdot U$. But only in that form we have evidence that four-currents are four-vectors. In our example we showed that total four-current J at least can be written as the sum of such four-currents. And the sum of four-vectors is a four-vector again as pointed out in **A2** .

C33 The Closed Loop in a Magnetic Field

In the laboratory frame S a metal rod is moving in x-direction at constant speed v . The rod is in conductive contact with a metal rail as shown in the figure below. Everywhere in the enclosed area we have a magnetic field pointing in the z-direction :



In the laboratory frame S a Lorentz force is acting on the electrons in the moving rod

$$\vec{f} = f_y = -e \cdot (-v \cdot B_z) = e \cdot v \cdot B_z \tag{33.1}$$

inducing an electric current I in the closed circuit. With Faraday's law we can calculate the induced voltage U :

$$U_{ind} = \left|\frac{d\Phi}{dt}\right| = B \cdot \frac{dA}{dt} = B \cdot l \cdot v$$

The induced current depends on the resistance R of the circuit :

$$I = U_{ind} \,/\, R = B \cdot l \cdot v \,/\, R$$

In the frame S' of the rod the average speed of the electrons is zero, and hence no Lorentz force can be at work. But in the transformed field a non-zero electric field component $E_y{}'$ shows up :

$$E_x{}' = E_x = 0 \quad , \quad E_y{}' = \gamma_v \cdot (E_y - v \cdot B_z) = -\gamma_v \cdot v \cdot B_z \quad , \quad E_z{}' = \gamma_v \cdot (E_z + v \cdot B_y) = 0$$

The electrons in the rod are subjected to a Coulomb force of amount

$$\vec{f}' = f_y{}' = (-e) \cdot (-\gamma_v) \cdot v \cdot B_z = \gamma_v \cdot e \cdot v \cdot B_z = \gamma_v \cdot \vec{f} \tag{33.2}$$

As in the previous sections we have $\vec{f}' = \gamma_v \cdot \vec{f}$. Following **C29** the induced current is greater by the same factor γ_v in S' , too :

$$I' = \gamma_v \cdot I \tag{33.3}$$

And what about the induced voltage in system S' ?

$$|U_{ind}{}'| = \frac{d\Phi'}{dt'} = B_z{}' \cdot \frac{dA'}{d\tau} = \gamma_v \cdot B_z \cdot \frac{dA/\gamma_v}{dt} \cdot \frac{dt}{d\tau} = \gamma_v \cdot B \cdot \frac{dA/\gamma_v}{dt} \cdot \gamma_v = \gamma_v \cdot B \cdot \frac{dA}{dt} = \gamma_v \cdot |U_{ind}|$$

Therefore, for Ohm's resistance we have $R' = R$:

$$R' = \frac{|U_{ind}{}'|}{I'} = \frac{\gamma_v \cdot |U_{ind}|}{\gamma_v \cdot I} = \frac{|U_{ind}|}{I} = R$$

C34 The First Invariant of the Electromagnetic Field

Let a frame S' move at constant speed v as seen from another frame S . Let some electromagnetic field be given by $\vec{E}$ and $\vec{B}$ in S and by $\vec{E}'$ and $\vec{B}'$ in S' . Then the following equation holds true

$$\vec{E} \cdot \vec{B} = \vec{E}' \cdot \vec{B}' \tag{34.1}$$

The inner product $\vec{E} \cdot \vec{B} = \vec{E}(t,x,y,z) \cdot \vec{B}(t,x,y,z)$ is relativistically invariant. Therefore, if $\vec{E}$ and $\vec{B}$ are perpendicular to each other in system S they are perpendicular to each other in any other inertial system S' .

We could prove (34.1) using the equations (27.1) as done in [2 - 34.20]. We prefere another way, introducing a matrix M which will turn out to be very useful in sections **C38** and **C39** :

$$M = \begin{pmatrix} 0 & c \cdot B_x & c \cdot B_y & c \cdot B_z \\ c \cdot B_x & 0 & -E_z & E_y \\ c \cdot B_y & E_z & 0 & -E_x \\ c \cdot B_z & -E_y & E_x & 0 \end{pmatrix} \tag{34.2}$$

M gives a 'dual' description of the electromagnetic field and is closely connected with our matrix F . A simple matrix multiplication shows that

$$M \cdot F = F \cdot M = c \cdot \vec{E} \cdot \vec{B} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = c \cdot \vec{E} \cdot \vec{B} \cdot Id_4 \tag{34.3}$$

The *trace* , that is the sum of the main diagonal elements of $M \cdot F$, is $4 \cdot c \cdot \vec{E} \cdot \vec{B}$.

If F is the description of the electromagnetic field in System S then, following (24.4), $F' = L \cdot F \cdot L^{-1}$ is the description of the same electromagnetic field in system S' . You can easily check that $M' = L \cdot M \cdot L^{-1}$ holds for the matrix M . So we have

$$F' = L \cdot F \cdot L^{-1} \qquad \text{and} \qquad M' = L \cdot M \cdot L^{-1} \tag{34.4}$$

and hence

$$4 \cdot c \cdot \vec{E} \cdot \vec{B} = \text{trace}(M \cdot F) = \text{trace}(L \cdot M \cdot L^{-1} \cdot L \cdot F \cdot L^{-1}) = \text{trace}(M' \cdot F') = 4 \cdot c \cdot \vec{E}' \cdot \vec{B}'$$

This is our proof of (34.1) .

By the way, for the determinants of the matrices F, F', M and M' we have

$$det(F) = det(F') = det(M) = det(M') = -c^2 \cdot \left(\vec{E} \cdot \vec{B}\right)^2 \tag{34.5}$$

If a non-zero electromagnetic field can be transformed into a pure $\vec{B}'$-field we have $\vec{E}' = 0$ and hence $\vec{E}' \cdot \vec{B}' = 0$ everywhere. According to (34.1) that means that $\vec{E} \cdot \vec{B}$ has to be zero, too. Of course we have the same situation if some electromagnetic field can be transformed in a pure $\vec{E}'$-field.

C35 The Second Invariant of the Electromagnetic Field

If some coordinate frame S' moves at constant speed v along the x-direction of another frame S then the following equation concerning the descriptions of the electromagnetic field in S respective S' holds :

$$E^2 - c^2 \cdot B^2 = E'^2 - c^2 \cdot B'^2 \qquad (35.1)$$

$E^2 = \vec{E} \cdot \vec{E}$ and $B^2 = \vec{B} \cdot \vec{B}$ denote the inner products of the 3d-field vectors.

For a proof of (35.1) we consider the determinants of the matrices $F + M$ and $F - M$. The calculation shows that

$$det(F + M) = det(F - M) = -(E^2 - c^2 \cdot B^2)^2 \qquad (35.2)$$

In any case we have for 4x4-matrices $det(F - M) = det(M - F)$. Hence we have

$$det(F + M) = det(F - M) = det(M - F) = det(-M - F) = -(E^2 - c^2 \cdot B^2)^2 \qquad (35.3)$$

The corresponding statement in S' is

$$det(F' + M') = det(F' - M') = det(M' - F') = det(-M' - F') = -(E'^2 - c^2 \cdot B'^2)^2 \qquad (35.4)$$

Further we know

$$\det(F + M) = \det(L \cdot (F + M) \cdot L^{-1}) = \det\big((L \cdot F + L \cdot M) \cdot L^{-1}\big) =$$
$$= \det\big((L \cdot F \cdot L^{-1} + L \cdot M \cdot L^{-1})\big) = det(F' + M')$$

We are not quite done with the proof of (35.1), but we know now that $(E^2 - c^2 \cdot B^2)^2 = \left(E'^2 - c^2 \cdot B'^2\right)^2$.
(25.1) shows that E^2 and B^2 are continuous functions of relative speed v . If the value of $E^2 - c^2 \cdot B^2$ is positive (e.g.) and if its absolute value is constant, then it stays positive with varying v , it cannot jump to $-(E^2 - c^2 \cdot B^2)$.
Therefore, not only the square of $E^2 - c^2 \cdot B^2$ is invariant, but $E^2 - c^2 \cdot B^2$ itself. With that (35.1) is proven.

Of course you can prove (35.1) by means of (27.1), showing directly that $E'^2 - c^2 \cdot B'^2$ equals $E^2 - c^2 \cdot B^2$. This calculation is done in [2 - 34.21] .

(35.1) shows the impossibility of turning a pure electric field $\vec{E}$ into a pure magnetic field $\vec{B}'$: Then we would have $E^2 = -c^2 \cdot B'^2$ which implies $E^2 = 0 = B'^2$. Further we learn from (35.1) that some electromagnetic field can only be transformed in a pure magnetic field if $E^2 - c^2 \cdot B^2 \leq 0$. And, similarly, an electromagnetic field can only be transformed in a pure electric field if $E^2 - c^2 \cdot B^2 \geq 0$. In both cases we have the additional condition of $\vec{E} \cdot \vec{B} = 0$ according to the prevous section.

The necessary and sufficient conditions for such a transformation are studied in the next section.

C36 Which Fields can be Transformed to Zero ?

The answer comes from the equations (27.1). They contain the necessary and sufficient conditions for the existence of a transformation leading to $\vec{E}' = 0$ or $\vec{B}' = 0$.

If we evaluate (27.1) with $\vec{E}' = 0$ we find

$$E_x = 0 \quad , \quad E_y - v \cdot B_z = 0 \quad \text{and} \quad E_z + v \cdot B_y = 0$$

and hence

$$E_x = 0 \quad , \quad E_y = v \cdot B_z \quad \text{and} \quad E_z = -v \cdot B_y \tag{36.1}$$

This is the necessary and sufficient condition for the possibility to eliminate the electric field by applying a Lorentz boost. The necessary condition of **C34**, namely $\vec{E} \cdot \vec{B} = 0$, is already fulfilled :

$$\vec{E} \cdot \vec{B} = E_x \cdot B_x + E_y \cdot B_y + E_z \cdot B_z = 0 \cdot B_x + v \cdot B_z \cdot B_y + \left(-v \cdot B_y\right) \cdot B_z = v \cdot B_z \cdot B_y - v \cdot B_z \cdot B_y = 0$$

The necessary condition found in **C35** is fulfilled, too :

$$E^2 - c^2 \cdot B^2 = 0 + v^2 \cdot {B_z}^2 + v^2 \cdot {B_y}^2 - c^2 \cdot ({B_x}^2 + {B_y}^2 + {B_z}^2) = -c^2 \cdot {B_x}^2 + (v^2 - c^2) \cdot ({B_y}^2 + {B_z}^2) \leq 0$$

In the same way we find the conditions for the possibility to eliminate the magnetic field. From $\vec{B}' = 0$ and (27.2) we get

$$B_x = 0 \quad , \quad B_y + \frac{v}{c^2} \cdot E_z = 0 \quad \text{and} \quad B_z - \frac{v}{c^2} \cdot E_y = 0$$

and hence

$$B_x = 0 \quad , \quad B_y = -\frac{v}{c^2} \cdot E_z \quad \text{and} \quad B_z = \frac{v}{c^2} \cdot E_y \tag{36.2}$$

With that, the necessary conditions of **C34** and **C35** , namely $\vec{E} \cdot \vec{B} = 0$ and $E^2 - c^2 \cdot B^2 \geq 0$, are fulfilled, too.

C37 The Nabla-Operator as a Four-Form

Maxwell's equations can be written in a nice and compact form by means of the following 4d-Nabla-operator :

$$N_i = \left(\frac{1}{c} \cdot \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \tag{37.1}$$

The goal of this section is to give proof of

$$N_i' = \left(\frac{1}{c} \cdot \frac{\partial}{\partial t'}, \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'} \right) = N_i \cdot L^{-1} \tag{37.2}$$

In other words: N_i transforms like a four-form, obeying (9.6) .

We write out the right side of (37.2) :

$$N_i \cdot L^{-1} = \left(\frac{1}{c} \cdot \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \begin{pmatrix} \gamma & \gamma \cdot \beta & 0 & 0 \\ \gamma \cdot \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left(\frac{1}{c} \cdot \gamma \cdot \frac{\partial}{\partial t} + \gamma \cdot \beta \cdot \frac{\partial}{\partial x}, \frac{1}{c} \cdot \gamma \cdot \beta \cdot \frac{\partial}{\partial t} + \gamma \cdot \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

We have to show that the product equals N_i'.

- we have $\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}$ because we always have $dy = dy'$ in our setting
- similarly we have $\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$
- Let f be an arbitrary function depending on the variable t' . The equations (1.1) show how t' itself depends on the variables t and x . Hence we have

  $$\frac{\partial f}{\partial t'} = \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial t'} + \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t'} = \frac{\partial f}{\partial t} \cdot \gamma + \frac{\partial f}{\partial x} \cdot \gamma \cdot \beta \cdot c$$

  Obviously we have

  $$\frac{1}{c} \cdot \frac{\partial}{\partial t'} = \frac{1}{c} \cdot \gamma \cdot \frac{\partial}{\partial t} + \gamma \cdot \beta \cdot \frac{\partial}{\partial x}$$

  what proves the statement (37.2) for the first component of N_i' .

- Similarly we show that (37.2) holds for the second component of N_i' :

  $$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial x'} + \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial x'} = \frac{\partial f}{\partial t} \cdot \gamma \cdot \beta \cdot \frac{1}{c} + \frac{\partial f}{\partial x} \cdot \gamma$$

  and hence

  $$\frac{\partial}{\partial x'} = \frac{1}{c} \cdot \gamma \cdot \beta \cdot \frac{\partial}{\partial t} + \gamma \cdot \frac{\partial}{\partial x}$$

  This is exactly what we got from our matrix multiplication $N_i \cdot L^{-1}$.

Our 4d-Nabla-operator transforms like a four-form.

C38 Maxwell's Equations for Empty Space

Maxwell's equation are frequently written with the 3d-Nabla-operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T$. He is understood as a vector used to build inner products or cross products with the electric or magnetic field vectors.

- $\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \cdot \rho$ is short for $\frac{\partial \vec{E}}{\partial x} + \frac{\partial \vec{E}}{\partial y} + \frac{\partial \vec{E}}{\partial z} = \frac{1}{\varepsilon_0} \cdot \rho = \mu_0 \cdot c^2 \cdot \rho$
  Sources of the electric field are electric charges.

- $\nabla \times \vec{B} = \mu_0 \cdot \left(\vec{j} + \varepsilon_0 \cdot \frac{\partial \vec{E}}{\partial t}\right)$ is the short form for
  $\left(\frac{\partial \vec{B}_z}{\partial y} - \frac{\partial \vec{B}_y}{\partial z}, \frac{\partial \vec{B}_x}{\partial z} - \frac{\partial \vec{B}_z}{\partial x}, \frac{\partial \vec{B}_y}{\partial x} - \frac{\partial \vec{B}_x}{\partial y}\right)^T = \mu_0 \cdot \left(j_x + \varepsilon_0 \cdot \frac{\partial \vec{E}_x}{\partial t}, j_y + \varepsilon_0 \cdot \frac{\partial \vec{E}_y}{\partial t}, j_z + \varepsilon_0 \cdot \frac{\partial \vec{E}_z}{\partial t}\right)^T$
  Curls in the magnetic field arise around electric currents and in varying electric fields. $\vec{j}$ stands for the 3d-current density vector.

Those 1+3 equations can be expressed with our 4d-Nabla-operator and the matrix F in a single matrix equation :

$$\left(\frac{1}{c} \cdot \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & c \cdot B_z & -c \cdot B_y \\ E_y & -c \cdot B_z & 0 & c \cdot B_x \\ E_z & c \cdot B_y & -c \cdot B_x & 0 \end{pmatrix} = c \cdot \mu_0 \cdot (c \cdot \rho, -j_x, -j_y, -j_z) \tag{38.1}$$

Using our abbreviations we find a very compact form of (38.1) :

$$N_i \cdot F = c \cdot \mu_0 \cdot J_i \tag{38.2}$$

where $J_i = (c \cdot \rho, -j_x, -j_y, -j_z)$ represents the four-form corresponding to the four-current J^i. Further we used the identity $\varepsilon_0 \cdot \mu_0 = 1/c^2$.

Now to the second half of Maxwell's equations.

- $\nabla \cdot \vec{B} = 0$ means $\frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{B}}{\partial y} + \frac{\partial \vec{B}}{\partial z} = 0$.
  There is no such thing as a magnetic monopole.

- $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ means $\left(\frac{\partial \vec{E}_z}{\partial y} - \frac{\partial \vec{E}_y}{\partial z}, \frac{\partial \vec{E}_x}{\partial z} - \frac{\partial \vec{E}_z}{\partial x}, \frac{\partial \vec{E}_y}{\partial x} - \frac{\partial \vec{E}_x}{\partial y}\right)^T = \left(-\frac{\partial \vec{B}_x}{\partial t}, -\frac{\partial \vec{B}_y}{\partial t}, -\frac{\partial \vec{B}_y}{\partial t}\right)^T$
  Curls in the electric field are caused by varying magnetic fields.

Those 1+3 equations can be expressed with our 4d-Nabla-operator and the matrix M in a single matrix equation :

$$\left(\frac{1}{c} \cdot \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \begin{pmatrix} 0 & c \cdot B_x & c \cdot B_y & c \cdot B_z \\ c \cdot B_x & 0 & -E_z & E_y \\ c \cdot B_y & E_z & 0 & -E_x \\ c \cdot B_z & -E_y & E_x & 0 \end{pmatrix} = (0, 0, 0, 0) \tag{38.3}$$

or, using our abbreviations

$$N_i \cdot M = (0, 0, 0, 0) \tag{38.4}$$

C39 Maxwell's Equations are Covariant

In the last section we wrote down Maxwell's 2·(3+1) equations in a very compact way :

$$N_i \cdot F \;=\; c \cdot \mu_0 \cdot J_i$$

and

$$N_i \cdot M \;=\; 0_i$$

with the four-forms $J_i \;= (c \cdot \rho, -j_x, -j_y, -j_z)$ and $0_i = (0, 0, 0, 0)$ and the matrices F and M .

We also know how to transform those matrices and four-forms if we switch from one reference frame S to another frame S' . With all that given it is easy to show that Maxwell's equations hold true in the same way in S' as they do in S :

$$\begin{aligned} & N_i \cdot M \;=\; 0_i \\ \Leftrightarrow \quad & N_i \cdot L^{-1} \cdot L \cdot M \;=\; 0_i \\ \Leftrightarrow \quad & (N_i \cdot L^{-1}) \cdot (L \cdot M \cdot L^{-1}) \;=\; 0_i \cdot L^{-1} \\ \Leftrightarrow \quad & N_i{}' \cdot M' \;=\; 0_i \end{aligned}$$

The proof for the other four equations is done in the same manner :

$$\begin{aligned} & N_i \cdot F \;=\; c \cdot \mu_0 \cdot J_i \\ \Leftrightarrow \quad & N_i \cdot L^{-1} \cdot L \cdot F \;=\; c \cdot \mu_0 \cdot J_i \\ \Leftrightarrow \quad & (N_i \cdot L^{-1}) \cdot (L \cdot F \cdot L^{-1}) \;=\; c \cdot \mu_0 \cdot (J_i \cdot L^{-1}) \\ \Leftrightarrow \quad & N_i{}' \cdot F' \;=\; c \cdot \mu_0 \cdot J_i{}' \end{aligned}$$

Here we reap the fruits of our preparatory work !

Maxwell's equations are *covariant*, they have the same form in any inertial frame of reference. Maxwell's theory of the electromagnetic field, the invariance of the electric charge, the Lorentz force law and STR fit in a perfect way.

C40 Some Cosmetics for the Electromagnetic Field

In the matrices F and M the electric and the magnetic field appear in a slightly asymmetric way. This could be fixed with a small change in the definition of the electric field. A bunch of other aesthetic advantages would come with that small change :

- Let us use the following new definition of the electric field : $\mathrm{E} := E/c$
  We do not change the definition of the magnetic field : $\mathrm{B} := B$
  Now, both fields are measured in the unit 'Tesla'

- Let us use the matrices $\mathrm{F} := F/c$ and $\mathrm{M} := M/c$ to desribe the electromagnetic field. The factor c disappears, and the duality of the matrices becomes evident :

$$\mathrm{F} = \begin{pmatrix} 0 & \mathrm{E}_x & \mathrm{E}_y & \mathrm{E}_z \\ \mathrm{E}_x & 0 & \mathrm{B}_z & -\mathrm{B}_y \\ \mathrm{E}_y & -\mathrm{B}_z & 0 & \mathrm{B}_x \\ \mathrm{E}_z & \mathrm{B}_y & -\mathrm{B}_x & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{M} = \begin{pmatrix} 0 & \mathrm{B}_x & \mathrm{B}_y & \mathrm{B}_z \\ \mathrm{B}_x & 0 & -\mathrm{E}_z & \mathrm{E}_y \\ \mathrm{B}_y & \mathrm{E}_z & 0 & -\mathrm{E}_x \\ \mathrm{B}_z & -\mathrm{E}_y & \mathrm{E}_x & 0 \end{pmatrix}$$

- The factor c disappears in the force law: $\mathrm{K} = \mathrm{q} \cdot \mathrm{F} \cdot \mathrm{U}$ instead of $K = \frac{q}{c} \cdot F \cdot U$

- The factor c disappears in the second half of Maxwells's eight equations :
  $\mathrm{N_i} \cdot \mathrm{F} = \mu_0 \cdot \mathrm{J_i}$ instead of $N_i \cdot F = c \cdot \mu_0 \cdot J_i$

- The factor c^2 disappears in the determinants of F and M :
  $det(\mathrm{M}) = det(\mathrm{F}) = -(\vec{\mathrm{E}} \cdot \vec{\mathrm{B}})^2$ instead of $det(M) = det(F) = -c^2 \cdot (\vec{E} \cdot \vec{B})^2$

- The factor c^2 disappears in the product of F and M :
  $\mathrm{F} \cdot \mathrm{M} = (\vec{\mathrm{E}} \cdot \vec{\mathrm{B}}) \cdot Id_4$ instead of $F \cdot M = c^2 \cdot (\vec{E} \cdot \vec{B}) \cdot Id_4$

- The factor c^2 disappears in the second invariant of the electromagnetic field :
  $\vec{\mathrm{E}}^2 - \vec{\mathrm{B}}^2$ instead of $\vec{E}^2 - c^2 \cdot \vec{B}^2$

- The set (25.1) of transformations of the electromagnetic field becomes symmetric in $\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$:

$$\begin{aligned} \mathrm{E_x}' &= \mathrm{E_x} & \qquad \mathrm{B_x}' &= \mathrm{B_x} \\ \mathrm{E_y}' &= \gamma_v \cdot (\mathrm{E_y} - \beta \cdot \mathrm{B_z}) & \qquad \mathrm{B_y}' &= \gamma_v \cdot (\mathrm{B_y} + \beta \cdot \mathrm{E_z}) \\ \mathrm{E_z}' &= \gamma_v \cdot (\mathrm{E_z} + \beta \cdot \mathrm{B_y}) & \qquad \mathrm{B_z}' &= \gamma_v \cdot (\mathrm{B_z} - \beta \cdot \mathrm{E_y}) \end{aligned}$$

- For the three-force $\vec{f}$ we now have $\vec{f} = q \cdot (c \cdot \vec{\mathrm{E}} + \vec{u} \times \vec{\mathrm{B}})$. Both field vectors get multiplied by a velocity.

All those simplifications would follow directly from giving the speed of light the value 1 , e.g. by measuring time in seconds and lengths in light-seconds. The speed of light would then be 'one lightsecond per second', that is 1 or '1 light'.

Centuries ago Carl Friederich Gauss has suggested an even more radical approach: Let the units of the electric and the magnetic field be defined so that the field constants ε_0 and μ_0 have the value 1 . Then the speed of light would be 1 , too, and the field constants would disappear together with the speed of light from Maxwell's equations.

Our new definition $\mathrm{E} := E/c$ is just the *most gentle* intervention to achieve the desired goal of symmetry between the electric and the magnetic field vectors.