Johannes Kepler and Robert Hooke

Kepler's contributions to the development of modern astronomy and physics are widely acknowledged. Far less appreciated are those of Robert Hooke. This is to a great part due to the mean behaviour of Isaac Newton. Only after Hooke's death in 1703 Newton agreed to become president of the Royal Society. His first actions as president were to destroy all instruments and papers left by Hooke after his 40 years of work for the Royal Society. Even Hooke's portrait went into the fire. So the Royal Society now owns a portrait of all of their presidents but of the first.

These informations are taken from the footnote on page 51 of the book "Huygens and Barrow, Newton & Hooke" by Vladimir I. Arnol'd (Birkhäuser Verlag 1990, translated into German and English from the Russian). **The present paper is nothing but an elaboration of appendix 1 of this book of Arnol'd !** It gives another, quite different and independent proof of Kepler's first law, and so it fits perfectly in my Kepler series.

Once again, Alfred Hepp has turned my manuscript and the many hand-made drawings into a perfect LaTeX document. I would like to express my thankfulness here for his careful and time-consuming work !

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1 Ellipses and Hooke's Law

Elastic behaviour is modelled in many cases in good approximation by Hooke's law:

$$\vec{F} = m \cdot \vec{a} = k \cdot \vec{r}$$

If some body gets attracted according to this law to the center of the coordinate system its trajectory is a solution of the following simple differential equation

$$\ddot{\boldsymbol{r}} = -\frac{k}{m} \cdot \boldsymbol{r} = -c \cdot \boldsymbol{r}$$

The solutions of this equation are ellipses centered at (0/0). Orientation and shape of the ellipse are given by the initial conditions $\vec{r}(t_0)$ and $\vec{v}(t_0)$. By these two vectors the plane of the trajectory is determined, too. By choice of the coordinate axes in this plane and the zeropoint of time the trajectories are given by the formulas

$$x = a \cdot \cos(\omega \cdot t), \;\; y = b \cdot \sin(\omega \cdot t)$$

Newton gives an elementary proof of that in his proposition X. You can easily realize such trajectories on an air cushion table. Hooke himself used as an analog machine a parabolic bowl like the satellite antennas of our days and a small ball rolling in that bowl.

Total energy of such a particle oscillating in two dimensions is given by

$$E_{\text{tot}} = 0.5 \cdot m \cdot v^2 + 0.5 \cdot k \cdot r^2 = 0.5 \cdot m \cdot (v^2 + c \cdot r^2)$$

In absence of friction, total energy is constant. Therefore the expression $2 \cdot E_{\text{tot}}/m = v^2 + c \cdot r^2$ is a constant of movement. We will use this in the proof of theorem 2 in section 5.

2 Central Forces and Kepler's Second Law

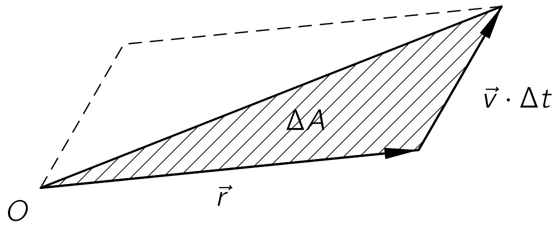
A force is called a central force, if it is everywhere directed towards the origin or straight away from the origin. Trajectories in a central force field lie in a plane. There is no component of the force driving the particle out of the plane given by $\vec{r}(t_0)$ and $\vec{v}(t_0)$.

If there is only a central force at work angular momentum of the trajectory is constant: The force is always parallel to the position vector, its lever is zero and hence it cannot generate any angular momentum. By a more mathematical argument: The vector $\vec{l} = \vec{r}(t) \times \vec{v}(t)$ is constant because its temporal derivative is zero. We have

$$\frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = \vec{0} + \vec{0} = \vec{0}$$

$\vec{l}$ is a constant vector, $\vec{L} = m \cdot \vec{l}$ is the angular momentum of the body on its trajectory.

Trajectories in a central force field always obey Kepler's second law. The radius vector sweeps out equal areas in equal time:



$$\Delta A = \frac{1}{2} \cdot |\vec{r} \times \vec{v} \cdot \Delta t| = $$
$$= \frac{1}{2} \cdot \Delta t \cdot |\vec{r} \times \vec{v}| = \frac{1}{2} \cdot \Delta t \cdot l$$

So we have

$$\frac{dA}{dt} = \frac{0.5 \cdot dt \cdot |\vec{r} \times \vec{v}|}{dt} = 0.5 \cdot |\vec{l}| = \text{konstant} = c_1$$

The detailled mathematics of the central force do not matter, Kepler's second law holds in any case. By the way: Kepler has first found his 'second' law, and later on he used it (as we will do) to find his 'first' law.

In polar coordinates we have in addition $dA = 0.5 \cdot r^2 \cdot d\varphi$. Hence we have for all movements in a central force field

$$\frac{0.5 \cdot r^2 \cdot d\varphi}{dt} = c_1$$

or

$$\frac{d\varphi}{dt} = \frac{c_2}{r^2}$$

3 The Mappings of Zhukovskii and Bohlin

Trajectories in a central force field lie in a plain. So they can be characterized in the plain of complex numbers. We define the following mappings of the complex plane onto itself:

$Squ(z) := z^2$	square function
$Zhu(z) := z + 1/z$	mapping of Zhukovskii
$Boh(z) := z^2 + 1/(z^2) + 2$	mapping of Bohlin
$Tr2(z) := z + 2$	translation by 2 to the right
$Srt(z) := \sqrt{r} \cdot cis(\varphi/2)$ where $z = r \cdot cis(\varphi)$	square root

The following propositions hold:

Lemma 1 $Boh = Squ \circ Zhu = Tr2 \circ Zhu \circ Squ$

Lemma 2 Any circle $|z| = r > 1$ is mapped by Zhu onto an ellipse with center at 0 and foci at +2 and −2.

Lemma 3 Any circle $|z| = r > 1$ is mapped by Boh onto an ellipse with center at 2 and foci at 0 and +4.

Lemma 4 The square function Squ maps Zhukovskii ellipses with center at 0 and foci at +2 and −2 onto Bohlin ellipses with center at 2 and foci at 0 and +4.

We will just give a proof for Lemma 2. Simple calculations show the correctness of the other statements.

Proof of lemma 2:

For $z = r \cdot \cos(\varphi) + i \cdot r \cdot \sin(\varphi)$ we have

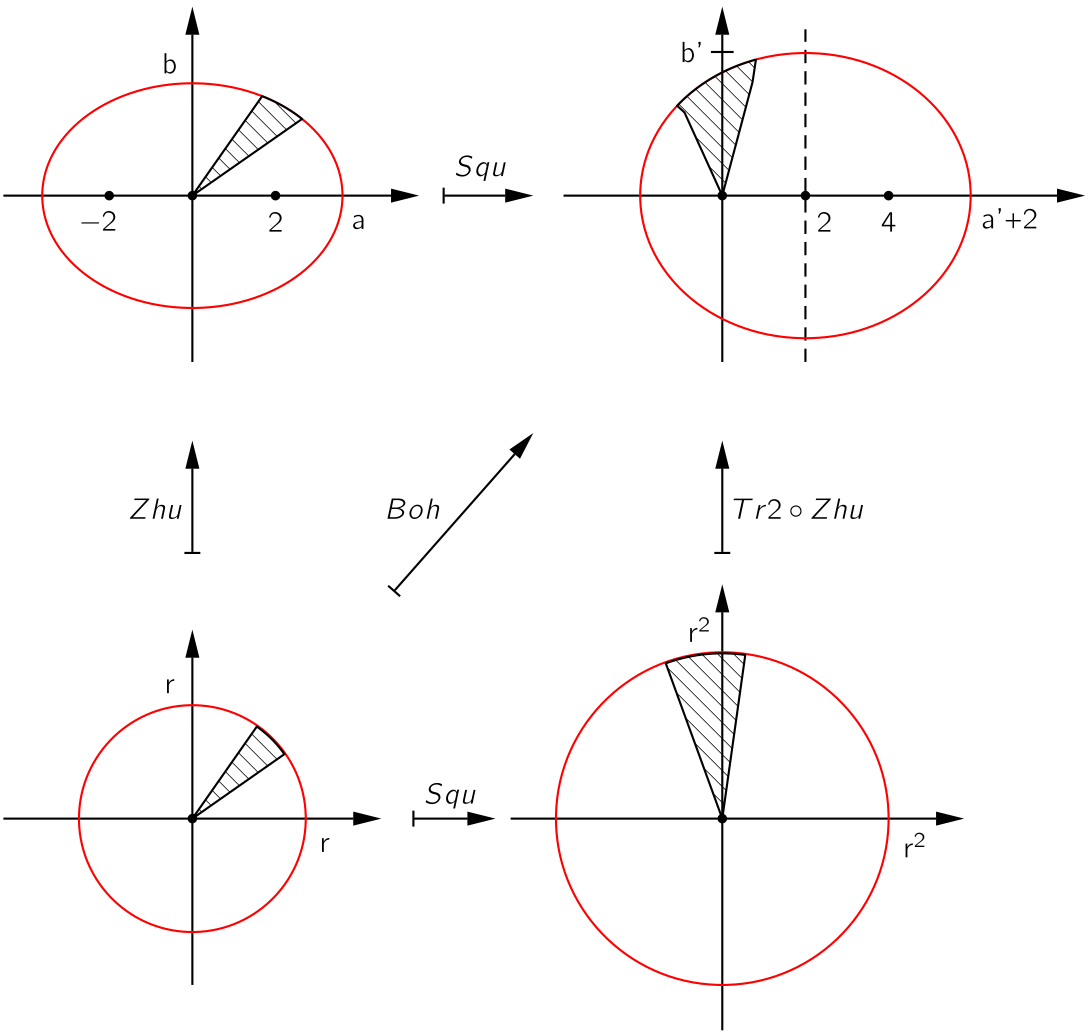
$$z + 1/z = r \cdot \cos(\varphi) + i \cdot r \cdot \sin(\varphi) + r^{-1} \cdot \cos(-\varphi) + i \cdot r^{-1} \cdot \sin(-\varphi) =$$
$$= (r + r^{-1}) \cdot \cos(\varphi) + i \cdot (r - r^{-1}) \cdot \sin(\varphi) = a \cdot \cos(\varphi) + i \cdot b \cdot \sin(\varphi)$$

The linear eccentricity is given by

$$c^2 = a^2 - b^2 = (r + r^{-1})^2 - (r - r^{-1})^2 = 2 - (-2) = 4$$

□

The above lemmata can be expressed by the statement that the following diagram commutes:



4 The Ellipses of Zhukovskii, Bohlin, Hooke and Kepler

Let me introduce the following names:

- Hooke ellipses are arbitrary ellipses with center in the origin
- Kepler ellipses are arbitrary ellipses with one focus in the origin
- Zhukovskii ellipses are ellipses with one focus in +2 and the other in -2
- Bohlin ellipses are ellipses with one focus in the origin and the other in +4

Lemma 5 There exists exactly one Zhukovskii ellipse and one Bohlin ellipse to each ratio $x = a : b > 1$ of semi-axes.

Proof: From $x = a : b$ and $a^2 - b^2 = 4$ we have $a = 2 \cdot x/\sqrt{x^2 - 1}$ and $b = 2/\sqrt{x^2 - 1}$. By $a = r + r^{-1}$ and $b = r - r^{-1}$ the radius $r = (a + b)/2$ is given by $r = (x + 1)/\sqrt{x^2 - 1}$. Zhu maps the circle with that radius onto the Zhukovskii ellipse with that ratio of semi-axes.

The same argument goes for Bohlin ellipses, just replace r by r^2.

Lemma 6 Each Hooke ellipse is the image of the Zhukovskii ellipse with the same ratio of semi-axes by a rotation around 0 and a dilatation; and each Kepler-ellipse is the image of the Bohlin ellipse with the same ratio of semi-axes by a rotation around 0 and a dilatation.

Proof: Choose the unique Zhukovskii ellipse with the same ratio of semi-axes as the given Hooke-ellipse. If the main axis of the Hooke ellipse has angle φ to the real axis of the coordinate system, and if its numerical eccentricity is c, then the rotation and the dilatation are given by $h = (c/2) \cdot cis(\varphi)$. Multiplication by h will map the Zhykovskii ellipse onto the given Hooke-elllipse.

The argument for Kepler ellipses follows the same line.

So for each Hooke ellipse there exists a single circle centered in 0 and radius $r > 1$ that will be mapped onto that Hooke ellipse by

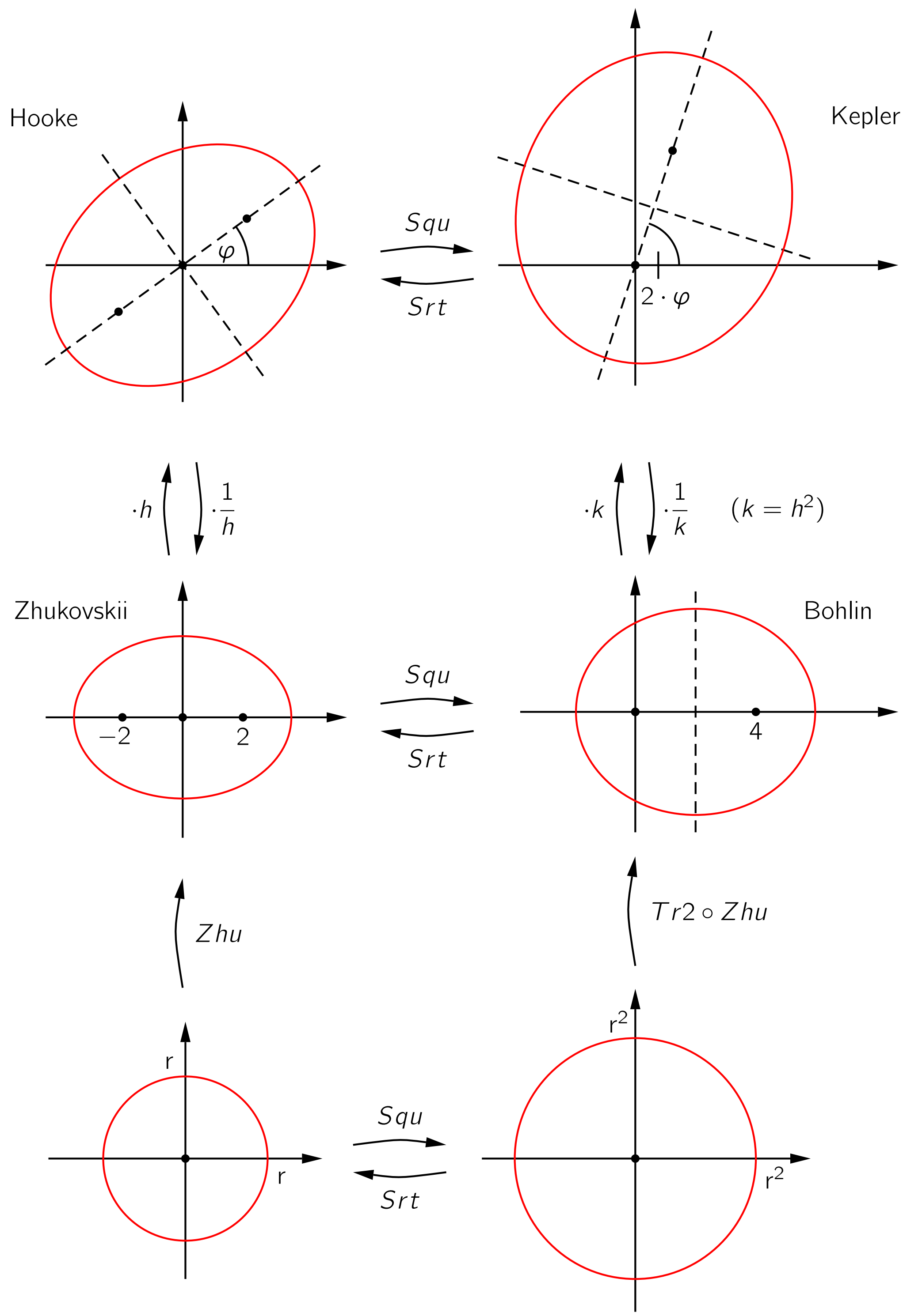
$$f(z) = h \cdot \left(z + \frac{1}{z}\right)$$

Lemma 5 and 6 tell us how to choose the radius r and the complex number h. The analogous statement holds for Kepler ellipses. We get the following

Theorem 1 The square of a Hooke ellipse is a Kepler ellipse, and each Kepler ellipse is the square of an unique Hooke ellipse.

Proof: Lemma 4 and Lemma 6. □

The following diagram illustrates what we have learnt so far. The square-function is not one to one, the image is covered twice. But as a whole the trajectories are clearly mapped onto each other.



5 Bohlin's Theorem

Let a particle or some body move in the complex plain according to Hooke's law.

$$\ddot{w} = -c \cdot w$$

The trajectory of the body is a Hooke ellipse if c is positive. Taking the square of this movement we get a Kepler ellipse (theorem 1). Bohlin's theorem says that this squared trajectory satisfies Newton's law:

Theorem 2 Let a pointlike mass mass move in the complex plain according to Hooke's law $\ddot{w} = -c \cdot w$. We take the square of the trajectory and introduce a new time τ for the points $z = w^2$ that satisfies Kepler's second law, too. Then $z(\tau)$ satisfies Netwon's law of gravitation

$$\frac{d^2 z}{d\tau^2} = -\frac{k \cdot z}{|z^3|}$$

Proof: Both times obey Kepler's second law. Hence we have according to section 2

$$\frac{d\varphi}{dt} = \frac{c_1}{|w|^2} \quad \text{and} \quad \frac{d(2 \cdot \varphi)}{d\tau} = \frac{c_2}{|z|^2}$$

Hence we have

$$\frac{\frac{d\varphi}{dt}}{\frac{2 \cdot d(\varphi)}{d\tau}} = \frac{d\tau}{2 \cdot dt} = \frac{\frac{c_1}{|w|^2}}{\frac{c_2}{|z|^2}} = c_3 \cdot \frac{|z|^2}{|w|^2} = c_3 \cdot \frac{|w|^4}{|w|^2} = c_3 \cdot |w|^2 = c_3 \cdot w \cdot \overline{w}$$

and

$$\frac{d\tau}{dt} = 2 \cdot c_3 \cdot w \cdot \overline{w} = c_4 \cdot w \cdot \overline{w}$$

or

$$\frac{1}{d\tau} = \frac{c_5}{w \cdot \overline{w}} \cdot \frac{1}{dt}$$

We will use that to replace derivations with respect to τ by derivations with respect to t.

We are free in choosing the speed of time τ. So we can manage the constant c_5 to be 1. The proof might as well be done without this simplification.

$$\frac{d^2z}{d\tau^2} = \frac{1}{w\cdot\overline{w}}\cdot\frac{d}{dt}\left(\frac{1}{w\cdot\overline{w}}\cdot\frac{d}{dt}(w^2)\right) = \frac{1}{w\cdot\overline{w}}\cdot\frac{d}{dt}\left(\frac{1}{w\cdot\overline{w}}\cdot\frac{2\cdot w\cdot dw}{dt}\right) =$$

$$\frac{2}{w\cdot\overline{w}}\cdot\frac{d}{dt}\left(\frac{1}{\overline{w}}\cdot\frac{dw}{dt}\right) = \frac{2}{w\cdot\overline{w}}\cdot\left(\frac{-1}{\overline{w}^2}\cdot\frac{d\overline{w}}{dt}\cdot\frac{dw}{dt}+\frac{1}{\overline{w}}\cdot\frac{d^2w}{dt^2}\right) =$$

$$\frac{2}{w\cdot\overline{w}}\cdot\left(\frac{-1}{\overline{w}^2}\cdot\frac{d\overline{w}}{dt}\cdot\frac{dw}{dt}+\frac{1}{\overline{w}}\cdot(-c\cdot w)\right) = \frac{-2}{w\cdot\overline{w}^3}\cdot\left(\frac{d\overline{w}}{dt}\cdot\frac{dw}{dt}+c\cdot w\cdot\overline{w}\right) =$$

$$\frac{-2}{w\cdot\overline{w}^3}\cdot(v^2+c\cdot|w|^2) = \frac{-2}{w\cdot\overline{w}^3}\cdot\frac{2\cdot E_{\text{tot}}}{m} = -k\cdot\frac{w^2}{w^3\cdot\overline{w}^3} = -k\cdot\frac{z}{|z|^3}$$

Total energy E_{tot} of the mass on its Hooke ellipse is definitely positive. Hence the constant k in the last term ist positive, too.

□

So, Kepler ellipses *are* trajectories in a $1/r^2$-force field. To all initial conditions $\vec{r}(t_0)$ and $\vec{v}(t_0)$ there exists such a Kepler ellipse. And this ellipse is the unique solution with this initial conditions because the solutions vary continously with the initial conditions.

The Swedish astronomer Karl Petrus Theodor Bohlin has published this theorem in 1911 in vol. 28 of "Bull. Astr.". I could not figure out the full name of this journal. Probably it is about the "Astronomische Nachrichten" of the Leibnitz-Institute in Potsdam, the oldest astronomical journal still existing nowadays. Since 1821 each year 10 volumes have been published. In 2022 the journal belongs to the Wiley-VCH publishing house.

The square function maps trajectories in a field of force where the force is proportional to the distance r to the center onto trajectories in a field of force where the force is proportional to r^{-2}. Those two force laws are in a way dual to each other, the duality is given by the square function respecive by the square root.

We further follow the the presentation of Arnol'd and show that this idea of 'dual laws' can be generalized for any exponents in the force law.

6 Generalization of Bohlin's Theorem

Theorem 3 Trajectories in a central force field with force proportional to r^α are mapped onto trajectories in a central force field with force proportional to r^A by the function $z = w^\gamma$ if both of the following conditions hold:

$$(\alpha + 3) \cdot (A + 3) = 4 \quad \text{and} \quad \gamma = (\alpha + 3)/2$$

Let us test this in the situation of theorem 2. With Hooke's law we have $\alpha = 1$. From the above conditions we get $A = -2$ and $\gamma = 2$. And, indeed, it is the square function that maps trajectories in a Hooke field onto trajectories in the Newton field. Starting with Newton's law we have $\alpha = -2$ and hence $A = 1$ and $\gamma = \frac{1}{2}$. The inverse mapping is given by the square root.

There are self-dual force fields for $\alpha = -1$ and $\alpha = -5$. $\alpha = -1$ is the trivial case, the mapping is given by the identity. With $\alpha = -5$ it is the reciprocal function that generates the dual trajectories.

The proof of theorem 3 follows exactly the same line as the proof of theorem 2:

The premises are the following:

$$z = w^\gamma \quad , \quad \frac{d^2 w}{dt^2} = -c_1 \cdot w \cdot |w|^{\alpha - 1} \quad , \quad \frac{d\varphi}{dt} = \frac{c_2}{|w|^2} \quad \text{and} \quad \frac{d(\gamma \cdot \varphi)}{d\tau} = \frac{c_3}{|z|^2}$$

On the one side we have

$$\frac{\frac{d\varphi}{dt}}{\frac{d(\gamma \cdot \varphi)}{d\tau}} = \frac{d\tau}{\gamma \cdot d\varphi} \cdot \frac{d\varphi}{dt} = \frac{1}{\gamma} \cdot \frac{d\tau}{d\varphi}$$

and on the other side

$$\frac{\frac{d\varphi}{dt}}{\frac{d(\gamma \cdot \varphi)}{d\tau}} = \frac{\frac{c_2}{|w|^2}}{\frac{c_3}{|z|^2}} = c_4 \cdot \frac{|w|^{2 \cdot \gamma}}{|w|^2} = c_4 \cdot |w|^{(\alpha+3)-2} = c_4 \cdot |w|^{\alpha+1}$$

and hence

$$\frac{1}{d\tau} = \frac{c_5}{|w|^{\alpha+1}} \cdot \frac{1}{dt}$$

where c_5 can be assumed to be equal to 1 (speed of time τ).

We have to show that for $z = w^\gamma = w^{(\alpha+3)/2}$ the following is true:

$$\frac{d^2 z}{d\tau^2} = -k \cdot z \cdot |z|^{A-1}$$

Total energy in the first force field is constant, and we have

$$\frac{2 \cdot E_{\text{tot}}}{m} = |\dot{w}|^2 + \frac{2 \cdot c}{\alpha + 1} \cdot |w|^{\alpha+1} = |\dot{w}|^2 + \frac{c}{\gamma - 1} \cdot |w|^{\alpha+1}$$

What follows are the same steps as in the proof of theorem 2:

$$\frac{d^2 z}{d\tau^2} = \frac{1}{|w|^{\alpha+1}} \cdot \frac{d}{dt}\left(\frac{1}{|w|^{\alpha+1}} \cdot \frac{d}{dt}\left(\frac{d(w^\gamma)}{dt}\right)\right) = \frac{1}{|w|^{\alpha+1}} \cdot \frac{d}{dt}\left(\frac{1}{|w|^{\alpha+1}} \cdot \gamma \cdot w^{\gamma-1} \cdot \frac{dw}{dt}\right) =$$

$$= \frac{\gamma}{|w|^{\alpha+1}} \cdot \frac{d}{dt}\left(\frac{1}{w^{(\alpha+1)/2} \cdot \overline{w}^{(\alpha+1)/2}} \cdot w^{(\alpha+1)/2} \cdot \frac{dw}{dt}\right) = \frac{\gamma}{|w|^{\alpha+1}} \cdot \frac{d}{dt}\left(\frac{1}{\overline{w}^{(\alpha+1)/2}} \cdot \frac{dw}{dt}\right) =$$

$$= \frac{\gamma}{|w|^{\alpha+1}}\left(-\left(\frac{\alpha+1}{2}\right) \cdot \frac{1}{\overline{w}^{(\alpha+1)/2-1}} \cdot \frac{dw}{dt} \cdot \frac{dw}{dt} + \frac{1}{\overline{w}^{(\alpha+1)/2}} \cdot \frac{d^2 w}{dt^2}\right) =$$

$$= \frac{\gamma}{|w|^{\alpha+1}}\left(-\left(\frac{\alpha+1}{2}\right) \cdot \frac{1}{\overline{w}^{(\alpha+1)/2-1}} \cdot v^2 + \frac{1}{\overline{w}^{(\alpha+1)/2}} \cdot (-c) \cdot w \cdot |w|^{\alpha-1}\right) =$$

$$= \frac{\gamma}{|w|^{\alpha+1}}\left((1-\gamma) \cdot \overline{w}^{-\frac{\alpha+3}{2}} \cdot v^2 + \overline{w}^{-\frac{\alpha+3}{2}} \cdot \overline{w} \cdot (-c) \cdot w \cdot |w|^{\alpha-1}\right) =$$

$$= \frac{\gamma \cdot \overline{w}^{-\frac{\alpha+3}{2}}}{|w|^{\alpha+1}}\left((1-\gamma) \cdot v^2 - c \cdot |w|^2 \cdot \frac{\gamma-1}{\gamma-1} \cdot |w|^{\alpha-1}\right) =$$

$$= \frac{\gamma \cdot (1-\gamma) \cdot \overline{w}^{-\frac{\alpha+3}{2}}}{|w|^{\alpha+1}}\left(|\dot{w}|^2 + \frac{c}{\gamma-1} \cdot |w|^{\alpha+1}\right) =$$

$$= \frac{\gamma \cdot (1-\gamma) \cdot \overline{w}^{-\frac{\alpha+3}{2}}}{|w|^{\alpha+1}}\left(\frac{2 \cdot E_{\text{tot}}}{m}\right) = -k \cdot \frac{1}{|w|^{\alpha+1} \cdot \overline{w}^{\frac{\alpha+3}{2}}}$$

We have to show that the final term is identical to $-k \cdot z \cdot |z|^{A-1}$:

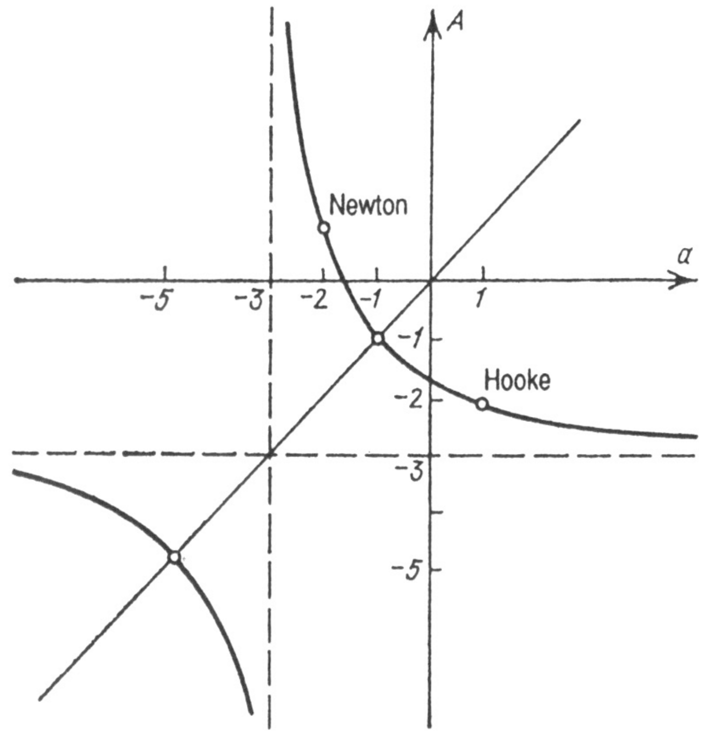
$$z \cdot |z|^{A-1} = w^\gamma \cdot \overline{w}^{\gamma \cdot (A-1)/2} \cdot w^{\gamma \cdot (A-1)/2} = w^\gamma \cdot \overline{w}^{1-2\cdot\gamma} \cdot w^{1-2\cdot\gamma} =$$

$$= w^{\frac{\alpha+3}{2}} \cdot \overline{w}^{-\alpha-2} \cdot w^{-\alpha-2} = w^{\frac{\alpha+3}{2}} \cdot |w|^{-2\cdot(\alpha+2)} = \frac{w^{\frac{\alpha+3}{2}} \cdot w^{\frac{\alpha+3}{2}}}{\overline{w}^{\frac{\alpha+3}{2}} \cdot |w|^{2\cdot(\alpha+2)}} =$$

$$= \frac{|w|^{\alpha+3}}{\overline{w}^{\frac{\alpha+3}{2}} \cdot |w|^{2\alpha+4}} = \frac{1}{\overline{w}^{\frac{\alpha+3}{2}} \cdot |w|^{\alpha+1}}$$

□

Arnol'd presents in a nice figure the symmetry underlying theorem 3. The dual pair of Hooke and Newton and the self-dual cases with exponents -1 and -5 are marked with a small circle.



Theorem 3 does not depend on the sign of the constant in the force law. Trajectories in a Hooke field where bodies are *repelled* with a force proportional to the distance are mapped on trajectories in a repelling Coulomb field or on trajectories with positive total energy in a Newton field. We will make this clear in the following section.

7 Trajectories in the Force Fields of Newton and Coulomb

For Newton's law with its exponent -2 theorem 3 yields

Corollary 1 All trajectories in Newton's gravitational field are mapped by the square root function onto trajectories in a central field of force where the force is proportional to the distance from the origin: $\ddot{w} = -c \cdot w$.

If the sign of the constant c is positive we are in the situation of the Hooke field, and Hooke ellipses are the corresponding trajectories. If the constant c is negative, the solutions are given by the hyperbolic functions

$$x = a \cdot coshyp(\omega \cdot t),\ \ y = b \cdot sinhyp(\omega \cdot t)$$

and the trajectories are hyperbolas centered in the origin (Hooke hyperbolas). If the constant is zero there is no force at work and the solutions are straight lines. There are but these three cases.

Kepler hyperbolas are, the other way round, squares of certain Hooke hyperbolas. If we are going to square Hooke hyperbolas we get Kepler hyperbolas with focus in the origin or Coulomb hyperbolas if the force is repulsing. If we square a straight line we get a Kepler parabola. Together with theorem 1 we get

Theorem 4 Trajectories in the Newton field of force are ellipses, hyperbolas or parabolas. Ellipses are the trajectories with negative total energy, hyperbolas belong to trajectories with positive total energy, and on parabolic trajectories total energy is zero.

Proof: As mentioned above only these three types of trajectories are possible. In the proof of theorem 3 we got

$$-k = \gamma \cdot (1 - \gamma) \cdot \left(\frac{2 \cdot E_{\text{tot}}}{m}\right)$$

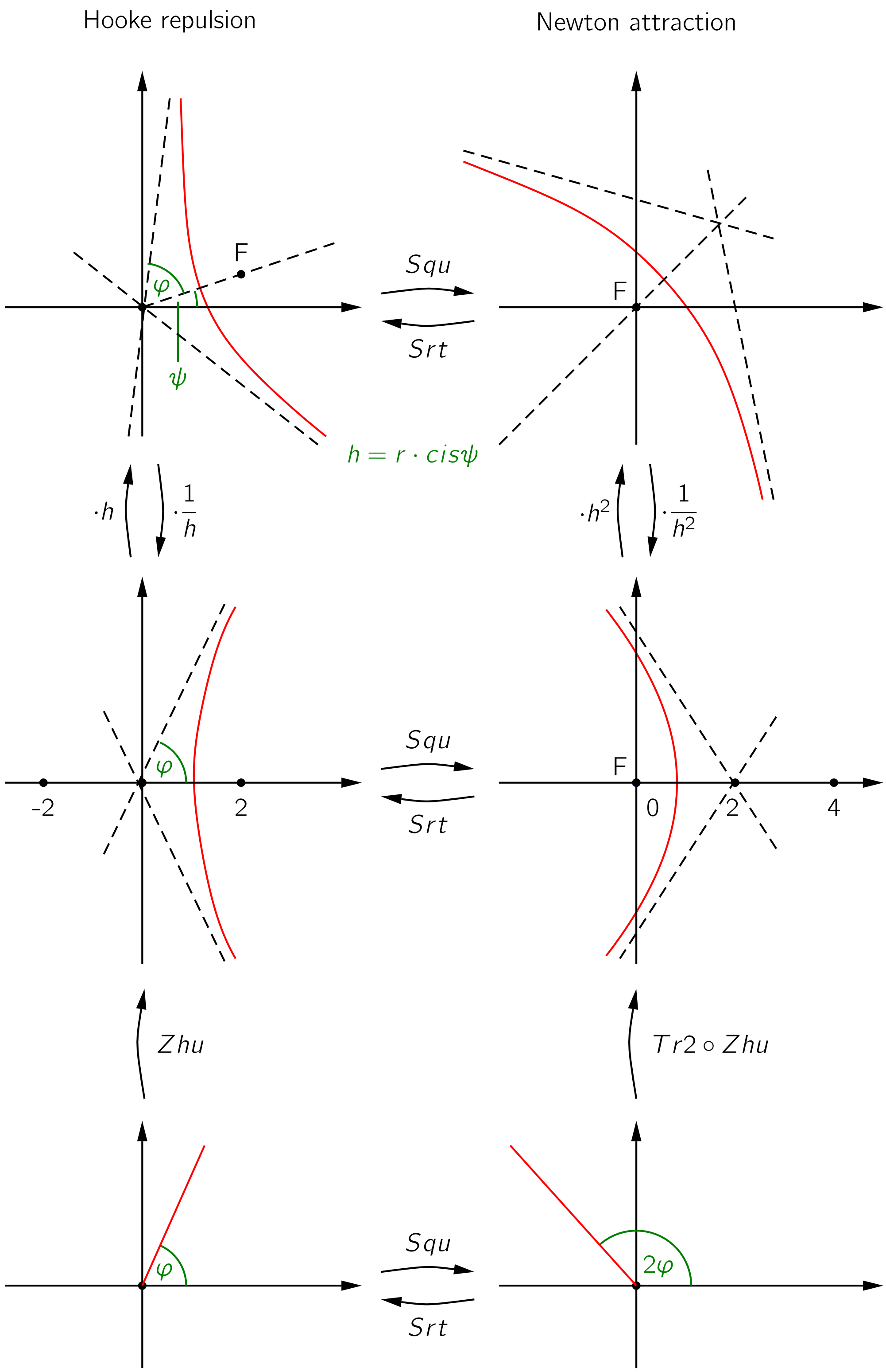
with

$$\alpha = -2,\ \ b = 1 \text{ and } \ \gamma = 1/2$$

The constant $-k$ in Hooke's force law has the same sign as the total energy of the body on its trajectory in the gravitational field, because $1 - \gamma = 1/2$ is positive. Hence the statements of theorem 4 are true.

□

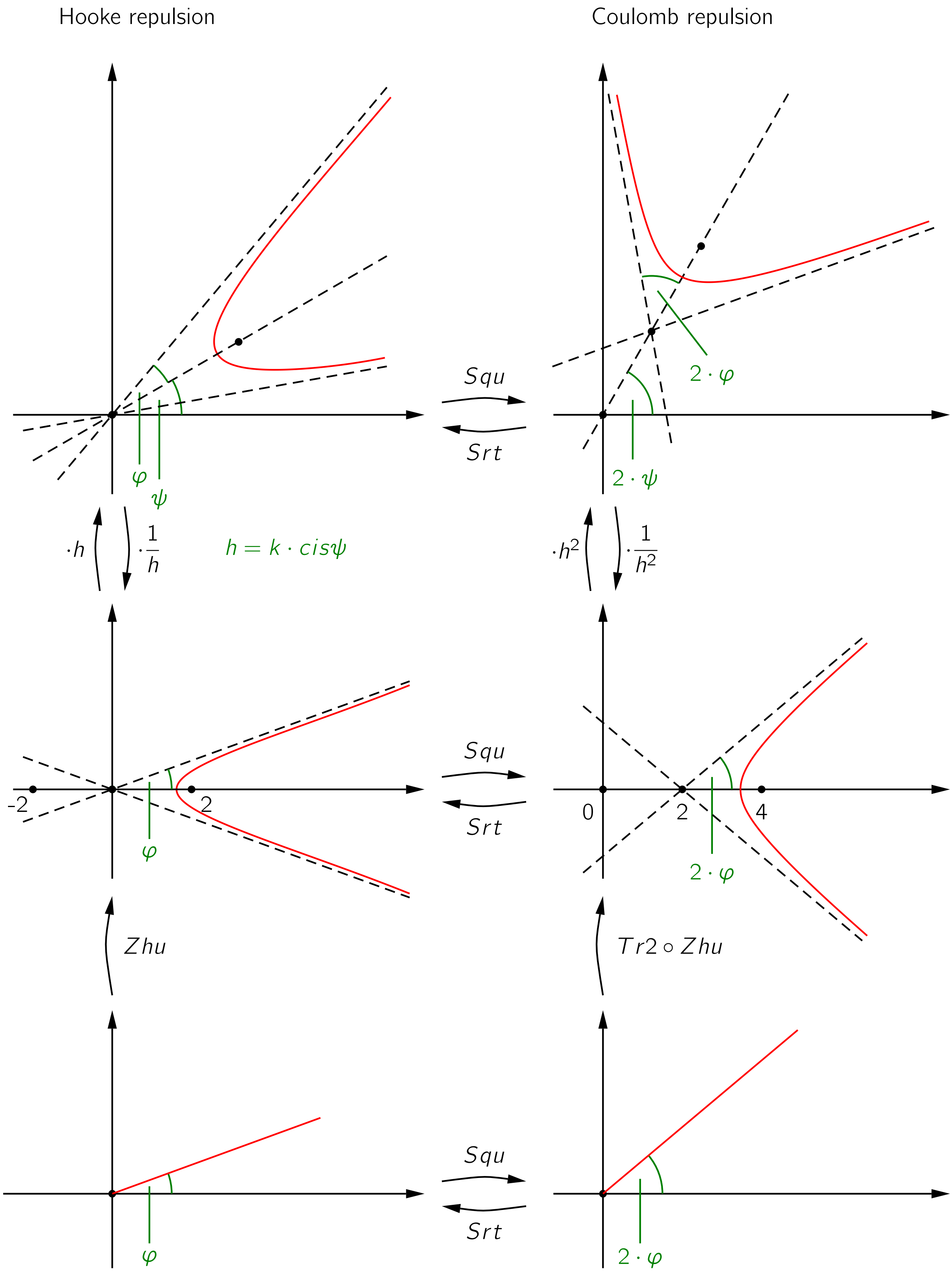
On the next pages we show how the hyperbolic trajectories in the force fields of Newton and Coulomb can be generated from straight lines by means of the Zhukovskii and the square function.



$45^\circ < \varphi < 90^\circ$!!

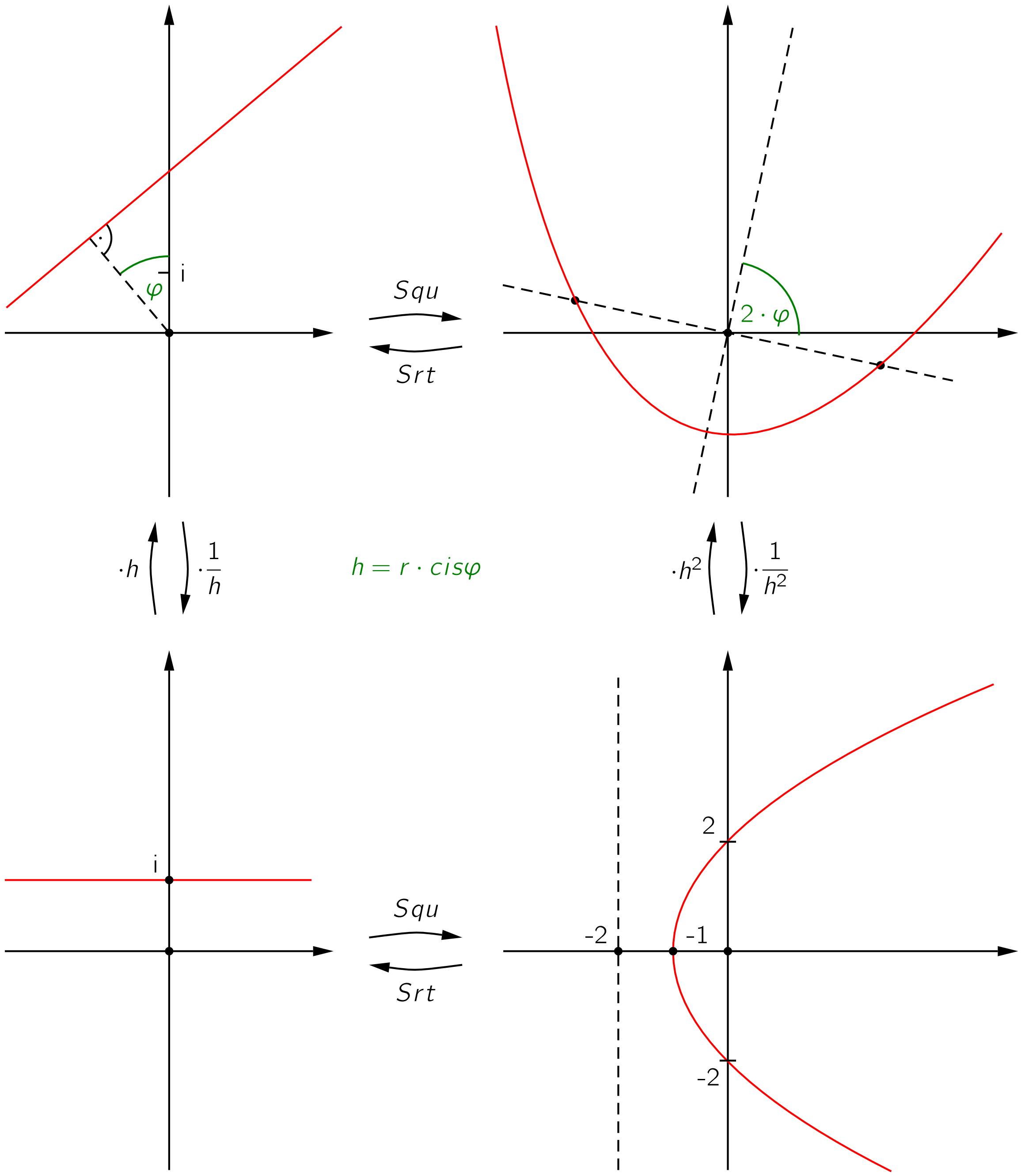
In the left row above and in the middle we have Hooke repulsion from the origin, in the right row we have Newton attraction to the origin. The angle φ has to take a value between 45° and 90° in order to get the desired result. For the other values compare the next page!

If the value of the angle φ lies between 0° and 45° we get hyperbolas, too. But they belong to a repulsive $1/r^2$- force law as might happen in the Coulomb case. These are the hyperbolas of alpha particles observed in Rutherford scattering.



$0° < \varphi < 45°$

Let us consider the third case with Hooke force zero. The straight lines in the zero Hooke field are mapped onto parabolas in the Newton or Coulomb field:



We can start with one single straight line because all parabolas are similar to each other.

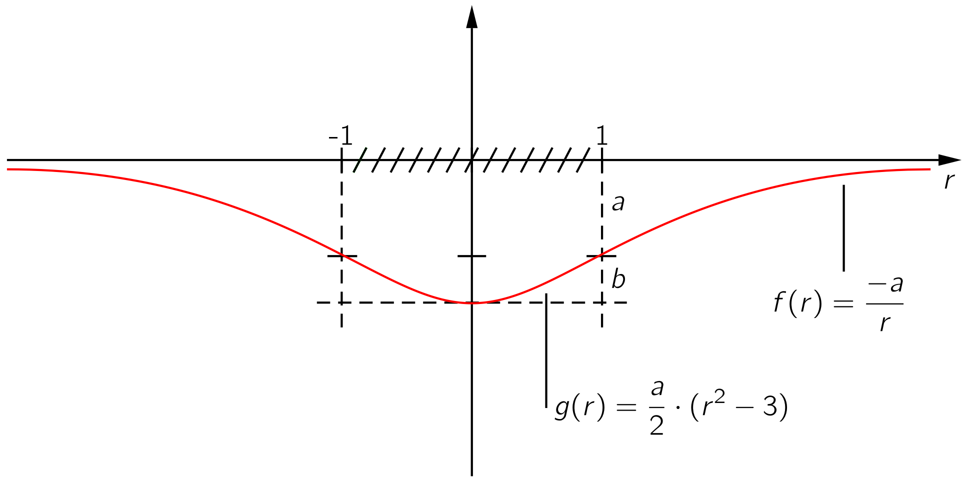
Theorema 2 and 3 include the cases of trajectories on straight lines through the origin, that is the center of force. These cases are especially simple to handle. In earlier days there was talking of ‘degenerated conic sections‘.

8 Hooke's Analog Machines

Robert Hooke suggested in a letter to Newton that the attracting force between massive bodies might fall with the square of the distance. Newton's reaction was to stop immediately exchanging mails with Hooke. With his supreme mathematical skills he succeeded to proove that in such a force field Kepler's laws must hold. The detailed elaboration 'more geometrico' led to his groundbreaking opus 'philosophiae naturalis principia mathematica', i.e. 'The Mathematical Principles of Natural Philosophy'.

Hooke tested his ideas experimentally. For different potentials of force fields he constructed surfaces to simulate the force field by means of gravitation. The potential to the Hooke field with $\ddot{\boldsymbol{r}} = -c \cdot \boldsymbol{r}$ is a parabola: $V(r) = b \cdot r^2$. If you let a ball roll in a parabolic satellite antenna you can observe Hooke's ellipses as the trajectory of the ball.

The potential to the $1/r^2$-force law is of type $V(r) = a \cdot r^{-1}$. The corresponding surface is generated by a hyperbola, rotating around the z-axis and approaching the x-y-plain. In several exhibitions you find such a funnel to illustrate ideas of general relativity (and to catch the coins you may let roll into that funnel). Combined with the Hooke field inside a homogenous spheric mass we get (without general relativity) the following 'analog machine':



Edmund Halley, acting as editor of the 'Principia', forced Newton to mention Hooke's contribution. So Newton conceded that *Hooke, Wren and Halley* had suggested the $1/r^2$-law before - only to avoid to give Hooke the deserved credits.