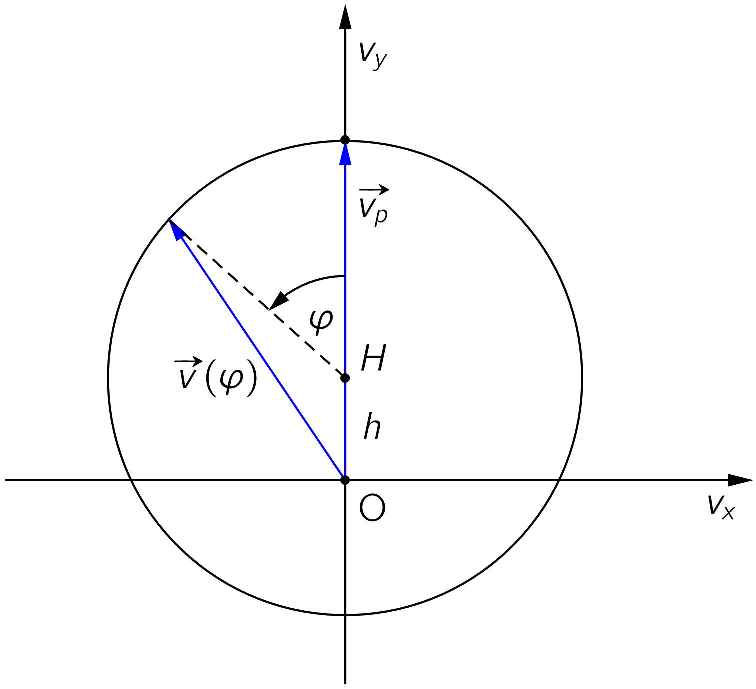
The Hodograph of an Elliptic Orbit

If you draw all velocity vectors of an orbit, the vectors always starting in the origin of the coordinate system, then the endpoints of those vectors will define the curve called "the hodograph of that orbit".

The hodograph of an elliptic orbit is a circle. We give a proof of that well-known result (Hamilton 1846) based only on Newton's law of gravity, the properties of ellipses and the law of cosines. Taking the formula for potential energy in the field of a spheric mass as given we do not need any calculus techniques.

The idea to this paper came from the lecture of "An algebra and trigonometry-based proof of Kepler's first law" by Akarsh Simha in Am. J. of Phys. 89 (11), November 2021.



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1 The Square of Velocity as a Function of Distance r

Following Newton, total energy E of a small mass m in the gravitational field of a huge mass M is given by

$$E = \frac{1}{2} \cdot m \cdot v^2 - \frac{G \cdot M \cdot m}{r} \tag{1}$$

We further suppose total energy to be negative. Akarsh Simha shows in the paper mentioned above that the small body then moves on an elliptic orbit around the big one.

The circular directrix (in German: Fallkreis $\sim$ fall circle) of this ellipse has radius $2 \cdot a$, where a is the semi-major axis of the ellipse. In distance $2 \cdot a$ of M the kinetic energy of the small body would be zero if total energy E remains constant. A proof of this fact is given in the paper of Akarsh Simha, too. So we have

$$E = \frac{1}{2} \cdot m \cdot v^2 - \frac{G \cdot M \cdot m}{r} = -\frac{G \cdot M \cdot m}{2 \cdot a} \tag{2}$$

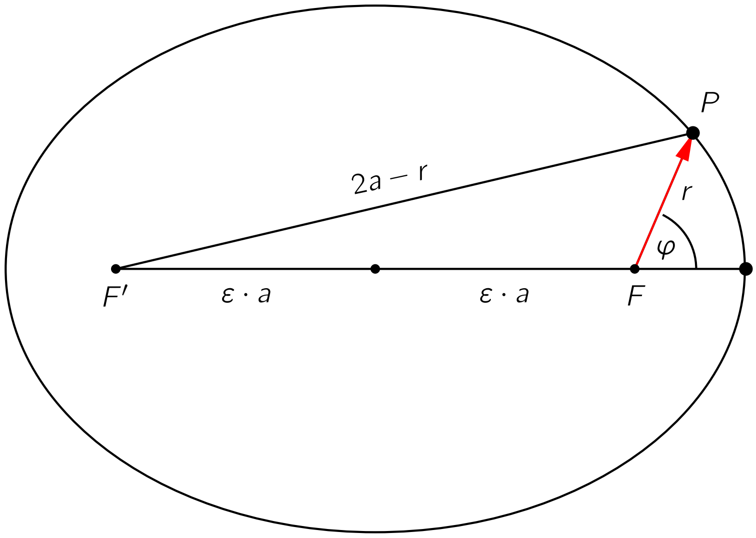
From that we get

$$v^2 = -\frac{G \cdot M}{a} + \frac{2 \cdot G \cdot M}{r} \tag{3}$$

In the next step we will replace the variable r in (3) by the angle φ between vector $\vec{r}$ and the vector pointing from M to the 'perihelion' of the orbit.

Read more about the 'fall circle' in "Kepler_09.pdf " on
https://www.physastromath.ch/material/mathematik/keplernewton/
(sorry, in German only)

2 The Square of Velocity as a Function of the angle φ



We have $F'P = 2 \cdot a - r$ and $F'F = 2 \cdot a \cdot \varepsilon$ if a denotes as usual the semi-major axis of the ellipse and ε stands for its eccentricity. The law of cosines tells us

$$(2 \cdot a - r)^2 = r^2 + (2 \cdot a \cdot \varepsilon)^2 - 2 \cdot r \cdot 2 \cdot a \cdot \varepsilon \cdot \cos(180^\circ - \varphi)$$

Simplified we get

$$a^2 \cdot (1 - \varepsilon^2) = r \cdot a \cdot (1 + \varepsilon \cdot \cos(\varphi))$$

and hence

$$r = \frac{a \cdot (1 - \varepsilon^2)}{1 + \varepsilon \cdot \cos(\varphi)} \tag{4}$$

(4) is the polar form of the ellipse; in the nominator we have the semi-latus rectum p .

Putting (4) into (3) we get the following, somewhat clumsy expression for v^2:

$$v^2 = -\frac{G \cdot M}{a} + \frac{2 \cdot G \cdot M \cdot (1 + \varepsilon \cdot \cos(\varphi))}{a \cdot (1 - \varepsilon^2)}$$

or

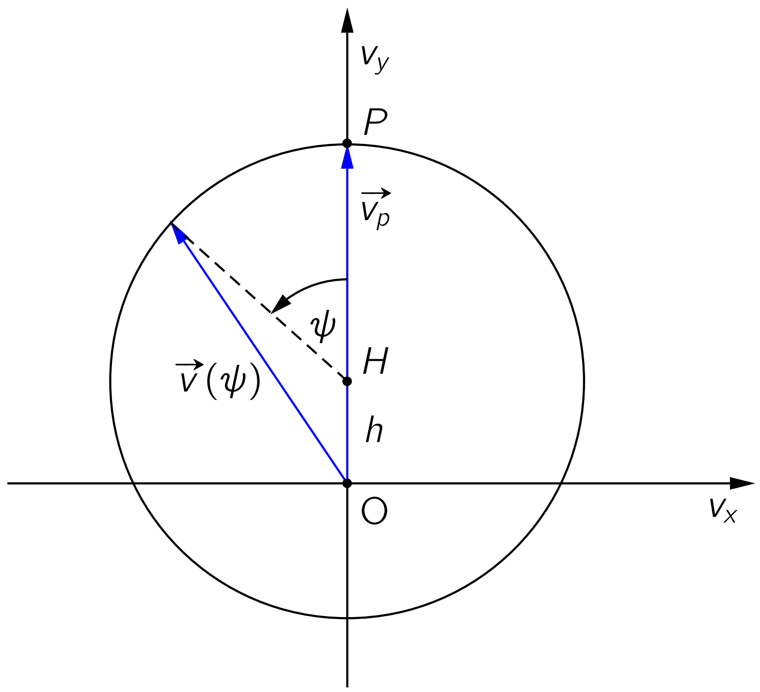
$$v^2 = \left[\frac{2 \cdot G \cdot M}{a \cdot (1 - \varepsilon^2)} - \frac{G \cdot M}{a}\right] + \left[\frac{\varepsilon \cdot 2 \cdot G \cdot M}{a \cdot (1 - \varepsilon^2)}\right] \cdot \cos(\varphi) \tag{5}$$

3 If the Hodograph would be a Circle ...

The hodograph is a circle if and only if there are two **constants** h and ρ so that for all velocities the following holds true:

$$v^2 = h^2 + \rho^2 - 2 \cdot h \cdot \rho \cdot \cos(180^\circ - \psi) = h^2 + \rho^2 + 2 \cdot h \cdot \rho \cdot \cos(\psi) \tag{6}$$

We have used the law of cosines once again. ρ denotes the radius of the circle, and $180^\circ - \psi$ is the angle opposite to the edge v in the triangle built with the edges v, h and ρ. The physical meaning of the angle ψ will turn out later.



$|\vec{v_p}| = h + \rho$ is the maximum speed of the small body in the point of minimal distance to the great body, that is in the 'perihelion' of the orbit.

In the next section we will calculate values for h and ρ that make the equations (5) and (6) become identical up to the names of the angles. To meet that goal the following equations must hold:

(i) $$h^2 + \rho^2 = \frac{2 \cdot G \cdot M}{a \cdot (1 - \varepsilon^2)} - \frac{G \cdot M}{a}$$

(ii) $$2 \cdot h \cdot \rho = \frac{\varepsilon \cdot 2 \cdot G \cdot M}{a(1 - \varepsilon^2)}$$

h and ρ enter the equations in a symmetric way. However, for an elliptic orbit h has to be smaller than ρ, because the velocity $\vec{v_a}$ in the 'aphelion' must be negative.

4 The Calculation of h and ρ

From other investigations we know that $h = \varepsilon \cdot \rho$ will hold. Using this as an *additional constraint* it is easy to calculate h and ρ. Substituting h by $\varepsilon \cdot \rho$ in equation (ii) yields

$$2 \cdot \varepsilon \cdot \rho^2 = 2 \cdot \varepsilon \cdot \frac{G\dot{M}}{a \cdot (1 - \varepsilon^2)}$$

and hence

$$\rho^2 = \frac{G \cdot M}{a \cdot (1 - \varepsilon^2)} \tag{7}$$

Simple calculations show then that (i) is fulfilled, too.

You do not like this trick? Then start with $h = x \cdot \rho$ and get from (i) and (ii) the quadratic equation

$$\varepsilon \cdot x^2 - (1 + \varepsilon^2) \cdot x + \varepsilon = 0$$

The solutions are $x = \varepsilon$ and $x = 1/\varepsilon$. But h has to be smaller than ρ, and hence we have to work with the solution $x = \varepsilon$.

Setting $\rho = \sqrt{\frac{G \cdot M}{a \cdot (1 - \varepsilon^2)}}$ and $h = \varepsilon \cdot \rho$ the equations (5) and (6) become identical up to the names of the angles. But if we have

$$h^2 + \rho^2 + 2 \cdot h \cdot \rho \cdot \cos(\varphi) = v^2 = h^2 + \rho^2 + 2 \cdot h \cdot \rho \cdot \cos(\psi)$$

we have $\cos(\varphi) = \cos(\psi)$ for all velocities and all angles $\varphi \in [0°, 360°]$.

The figure in section 3 tells us $v_x = -\rho \cdot \sin(\psi)$ and $v_y = h + \rho \cdot \cos(\psi)$. Together with $\cos(\varphi) = \cos(\psi)$ and the figure in section 2 the only possibility left is

$$\psi = \varphi \tag{8}$$

5 Summary

The hodograph of the bounded trajectory of a small mass in the gravitational field of a heavy mass is a circle.

The following statements are true:

$$E_{tot} = -\frac{G \cdot M \cdot m}{2 \cdot a}$$

$$r(\varphi) = \frac{a \cdot (1 - \varepsilon^2)}{1 + \varepsilon \cdot \cos(\varphi)} = \frac{p}{1 + \varepsilon \cdot \cos(\varphi)}$$

$$\rho = \sqrt{\frac{G \cdot M}{a \cdot (1 - \varepsilon^2)}} = \sqrt{\frac{G \cdot M}{p}}$$

$$h = \varepsilon \cdot \rho$$

$$v^2 = h^2 + \rho^2 + 2 \cdot h \cdot \rho \cdot \cos(\varphi)$$

$$\vec{v}(\varphi) = \begin{pmatrix} 0 \\ h \end{pmatrix} + \rho \cdot \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \end{pmatrix}$$

Be careful not to confuse the semi-latus rectum p with the radius ρ of the hodograph.

Only general properties of ellipses and the law of cosines have been used to derive the results. No calculus techniques are needed, if the formula for potential energy in the force field of Newton is accepted as given.