From Newton via Hamilton to Kepler

Another version of “One Newton yields three Kepler", based on a paper of Erich Ch. Wittman and the earlier papers of Kepler_0x.pdf.

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$$\frac{d\varphi}{dt} = \frac{w}{r^2}$$

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1 Orbits in a Central Force Field

If the acting force is everywhere directed to or away from a central point the force is called a central force. The central point is chosen as the zero point of the coordinate frame, and we have $\vec{r} \parallel \vec{a}$ or

$$\vec{r} \times \vec{a} = \vec{0} \tag{1}$$

Orbits in the field of a central force lie in a plane. The force acts always in the plane given by the origin and the initial vectors $\vec{r}(t_0)$ and $\vec{v}(t_0)$.

For any movement in the field of a central force we have

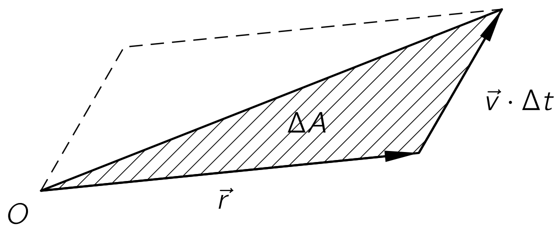
$$\vec{r} \times \vec{v} = \vec{w} = \text{konstant} \tag{2}$$

Using (1) it is easy to show that the derivative of $\vec{w}$ with respect to time is zero.

$\vec{w}$ is normal to the orbital plane. $\vec{w}$ and $w = |\vec{w}|$ are important *invariants* of the orbital motion.

$\vec{r} \times (m \cdot \vec{v}) = m \cdot (\vec{r} \times \vec{v}) = m \cdot \vec{w}$ is the *angular momentum* of the orbiting mass. In a central force field angular momentum is conserved.

2 The Area sweapt out by the Position Vector



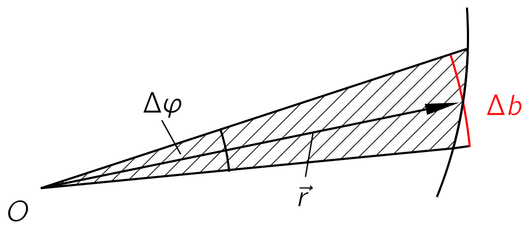
$$\Delta A = \tfrac{1}{2} \cdot |\vec{r} \times \vec{v} \cdot \Delta t| = \\ = \tfrac{1}{2} \cdot \Delta t \cdot |\vec{r} \times \vec{v}| = \tfrac{1}{2} \cdot \Delta t \cdot w$$

According to (2) we have in any central force field

$$\frac{dA}{dt} = \frac{1}{2} \cdot w = c = \text{constant} \tag{3}$$

Constant c is only introduced for better comparison with other scripts. (3) is the essence of Kepler's Second Law.

Trajectories in a plane can be expressed in polar coordinates:



$$\Delta A = \tfrac{1}{2} \cdot r \cdot \Delta b = \tfrac{1}{2} \cdot r \cdot r \cdot \Delta\varphi$$

For all planar trajectories we have without any further premise

$$\frac{dA}{d\varphi} = \frac{1}{2} \cdot r^2 \tag{4}$$

From (3) and (4) we get using the chain rule

$$\frac{d\varphi}{dt} = \frac{w}{r^2} \quad \text{and} \quad \frac{dt}{d\varphi} = \frac{r^2}{w} \tag{5}$$

Proof: The chain rule states

$$\frac{dA}{dt} = \frac{dA}{d\varphi} \cdot \frac{d\varphi}{dt}$$

and hence

$$\frac{1}{2} \cdot w = \frac{1}{2} \cdot r^2 \cdot \frac{d\varphi}{dt} \quad \text{and} \quad \frac{w}{r^2} = \frac{d\varphi}{dt} \qquad \square$$

3 Newton's Law of Gravitation

Following Newton we assume the central force to by spherically symmetric, and the absolute value of the force should be proportional to $\frac{1}{r^2}$:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{k}{r^2} \cdot \frac{-\vec{r}}{r} \tag{6}$$

If a small mass m moves in the gravitational field of a huge mass M we have

$$k = G \cdot M \tag{7}$$

G denoting Newton's gravitational constant.

In a spherically symmetric central force field not only angular momentum but also energy is conserved. We will need this fact in section 8.

4 The Hodograph lies on a Circle

According to (6) we have

$$\frac{d\vec{v}}{dt} = \frac{k}{r^2} \cdot \frac{-\vec{r}}{r} = \frac{k}{r^2} \cdot \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix}$$

Using the chain rule and (5) we get

$$\frac{d\vec{v}}{d\varphi} = \frac{d\vec{v}}{dt} \cdot \frac{dt}{d\varphi} = \frac{k}{r^2} \cdot \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix} \cdot \frac{r^2}{w} = \frac{k}{w} \cdot \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix} \tag{8}$$

Integration with respect to φ gives us the hodograph of the movement:

$$\vec{v}(\varphi) = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \frac{k}{w} \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

Constant h_1 is set to zero if we choose the coordinate frame so that $\vec{r}$ points to the perihelion of the trajectory for $\varphi = 0$. $\vec{r}$ reaches its minimal value only if $\vec{v}$ is orthogonal to $\vec{r}$. Then we can write

$$\vec{v}(\varphi) = \begin{pmatrix} 0 \\ h \end{pmatrix} + \frac{k}{w} \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \tag{9}$$

(9) is the equation of a circle with center point $H = (0/h)$ and radius ρ where

$$\rho = \frac{k}{w} \tag{10}$$

The constant of integration h will be calculated in section 7 and 8.

The ideas of the sections 4 and 5 originate from the following beautyful publication of Erich Ch. Wittmann:

“Von den Hüllkurvenkonstruktionen der Kegelschnitte zu den Planetenbahnen"
Mathematische Semesterberichte (2015) 62: 17-35, Springer Verlag 2015

5 Orbits in the Field of a Central Mass are Conic Sections

We get another equation for $\vec{v}$ caculating the derivative of

$$\vec{r} = r \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

with respect to time using (5) and the chain rule:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{d\varphi} \cdot \frac{d\varphi}{dt} = \frac{w}{r^2} \cdot \frac{d}{d\varphi}\left(r \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \right) = \frac{w}{r^2} \cdot \left(\frac{dr}{d\varphi} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} + r \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \right) \tag{11}$$

(9) and (11) both give a representation of $\vec{v}(\varphi)$. For the components of $\vec{v}(\varphi)$ we get

$$\text{I} \qquad 0 + \frac{k}{w} \cdot (-\sin\varphi) = \frac{w}{r^2} \cdot \frac{dr}{d\varphi} \cdot \cos\varphi + \frac{w}{r} \cdot (-\sin\varphi)$$

$$\text{II} \qquad h + \frac{k}{w} \cdot \cos\varphi = \frac{w}{r^2} \cdot \frac{dr}{d\varphi} \cdot \sin\varphi + \frac{w}{r} \cdot \cos\varphi$$

Multiplying I by $(-\sin\varphi)$ and II by $\cos\varphi$ and summation of the new equations yields

$$h \cdot \cos\varphi + \frac{k}{w} = \frac{w}{r}$$

and, multiplying by $r \cdot w/k$

$$r \cdot \left(\frac{h \cdot w}{k} \cdot \cos\varphi + 1 \right) = \frac{w^2}{k} \tag{12}$$

Defining

$$p = \frac{w^2}{k} \tag{13}$$

and

$$\varepsilon = \frac{h \cdot w}{k} \tag{14}$$

we get from (12) the equation of a conic section in polar coordinates:

$$r = \frac{p}{1 + \varepsilon \cdot \cos\varphi} \tag{15}$$

The orbits of the planets are ellipses with the sun in one of their focal points.

6 On the Proportions of the semi-major Axes and the Periods

In order to derive Kepler's third law we integrate (3) over a complete period T :

$$c \cdot T = \pi \cdot a \cdot b$$

Squared

$$c^2 \cdot T^2 = \pi^2 \cdot a^2 \cdot b^2$$

Using $b^2 = a \cdot p$, (13) and $w^2 = 4 \cdot c^2$ we get

$$\frac{a^3}{T^2} = \frac{c^2}{p \cdot \pi^2} = \frac{c^2 \cdot k}{4 \cdot c^2 \cdot \pi^2} = \frac{k}{4 \cdot \pi^2} = \frac{G \cdot M}{4 \cdot \pi^2} \tag{16}$$

The quotient a^3/T^2 takes the same value for all planets. We have got Kepler's third law.

Now, Newton's third law states that all forces between bodies are *interactions*, abbreviated oftly by *actio = reactio*. The sun has to move around the common center of mass of M und m , too. This leeds to a small correction of (16). The exact version of Kepler's third law within Newton's theory of gravitation is given by

$$\frac{a^3}{T^2} = \frac{G \cdot (M + m)}{4 \cdot \pi^2} \tag{17}$$

(17) is symmetric as M and m are concerned. In order to calculate the masses of the components of a binary star you have to use (17). The formula is derived in Kepler_01.pdf. Within our solar system (17) is a very small refinement of (16).

All the papers Kepler_xy.pdf are offered for download at

www.physastromath.ch/material/mathematik/keplernewton/

7 Excentricity, total Energy and the Center of the Hodograph

With (10) and (13) we have for the radius of the hodograph

$$\rho = \frac{k}{w} = \frac{G \cdot M}{w} = \frac{w}{p} \tag{18}$$

From (14) we get for the constant of integration h

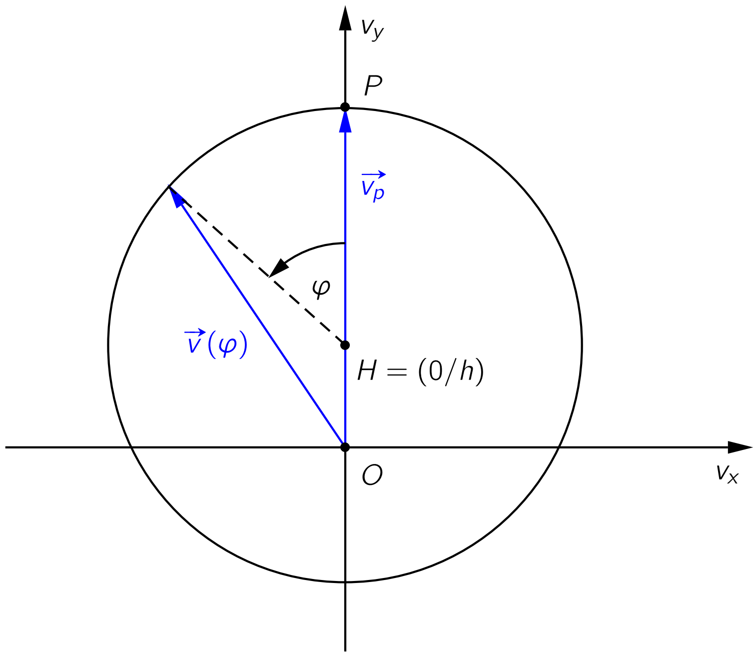
$$h = \varepsilon \cdot \frac{k}{w} = \varepsilon \cdot \rho \tag{19}$$

So we have

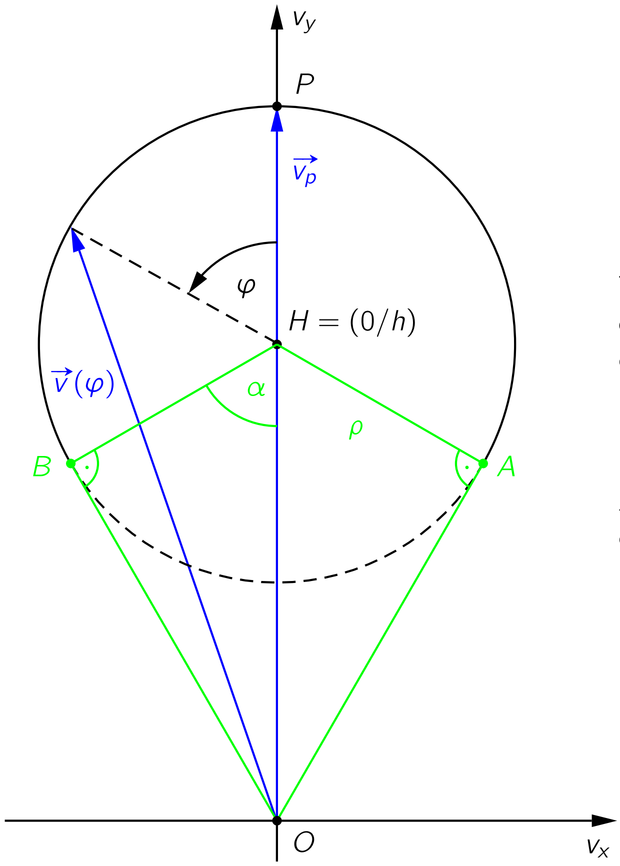
Orbit	Ellipse	Parabola	Hyperbola
Excenticity	$\varepsilon < 1$	$\varepsilon = 1$	$\varepsilon > 1$
total Energy	$E_{\text{tot}} < 0$	$E_{\text{tot}} = 0$	$E_{\text{tot}} > 0$
ρ and h	$h < \rho$	$h = \rho$	$h > \rho$
ρ and v_p	$v_p < 2 \cdot \rho$	$v_p = 2 \cdot \rho$	$v_p > 2 \cdot \rho$
Position of O	in the hodograph	on the hodograph	outside of the hodograph

Let's draw the hodographs in all three cases. $\vec{v}_p = \overrightarrow{OP}$ is the velocity of m in the perihelion, that is the maximum velocty of m.

For $\varepsilon < 1$ the center O is within the hodograph:



For $\varepsilon > 1$ the center O is exterior to the circle of the hodograph:



The tips of the velocity vectors all lie on the arc APB. A and B are reached at distance $r = \infty$ from the sun only.

Let's prove the following small proposition:

A and B are points of the Thales circle with diameter $\overline{OH}$.

Proof: $\varphi_{\max}$ is reached at $r = \infty$. Then we have according to (15)

$$1 + \varepsilon \cdot \cos\varphi_{\max} = 0$$

rearranged

$$\frac{1}{\varepsilon} = -\cos\varphi_{\max}$$

or

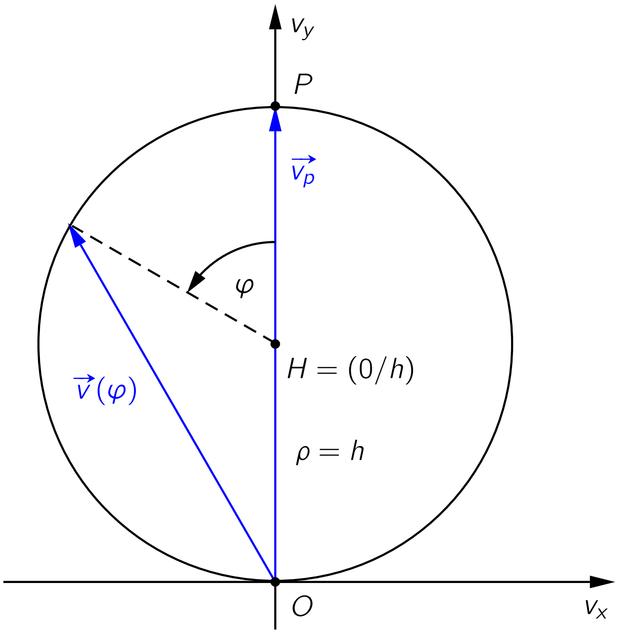
$$\frac{\rho}{h} = -\cos\varphi_{\max} = \cos(180^\circ - \varphi_{\max})$$

This holds if and only if A and B are points of the Thales circle with diameter $\overline{OH}$:

$$\cos(180^\circ - \varphi_{\max}) = \cos\alpha = \frac{\overline{BH}}{\overline{OH}} = \frac{\rho}{h}$$

□

For $\varepsilon = 1$ the hodograph looks as follows:



All points of the circle are covered but O. Associated to O is the distance $r = \infty$ from the sun. $|\overrightarrow{OP}| = 2 \cdot \rho$ is the maximum velocity in the perihelion.

8 Calculating the Orbital Elements from $\vec{r}(t_0)$ and $\vec{v}(t_0)$

Let $\vec{r}$ and $\vec{v}$ be known for any specific moment. Then we get from (2) the constants w and $c = \frac{1}{2} \cdot w$.

The central mass M gives us $k = G \cdot M$. With (10) we get the radius ρ of the hodograph, and from (13) we know the semi-latus rectum p of the orbit.

Now we use once more Newton's law of gravitation. The gravitational field of M is "conservativ", total energy is conserved:

$$\frac{1}{2} \cdot \cancel{m} \cdot v^2 - G \cdot M \cdot \cancel{m} \cdot \frac{1}{r} = E_{\text{tot}} = -G \cdot M \cdot \cancel{m} \cdot \frac{1}{2 \cdot a} \tag{20}$$

or

$$v^2 - \frac{2 \cdot k}{r} = -\frac{k}{a} \tag{21}$$

$2 \cdot a$ is the radius of the "cercle directeur" (in German: "Leitkreis") of the conic section (15). In case of an ellipse a is the semi-major axis of the ellipse! You find more on this in Kepler_09.pdf !

Solving (21) with respect to a we get

$$a = \frac{k \cdot r}{2 \cdot k - v^2 \cdot r} \tag{22}$$

The values of ε and h are still missing. (19) tells us that knowing ε means knowing h too. With $p = a \cdot (1 - \varepsilon^2)$ and (13) we get from (22)

$$\varepsilon^2 = 1 - \frac{p}{a} = 1 - \frac{2 \cdot w^2}{k \cdot r} + \frac{v^2 \cdot w^2}{k^2} \tag{23}$$

(23) determines ε and $h = \varepsilon \cdot \rho$.

(20) to (23) are equally valid for hyperbolas, with a having a negative sign. For parabolas a gets the value infinity from (20). However, for parabolas we always have $\varepsilon = 1$!

Using (10) and (13) we can rewrite (23) as follows:

$$\varepsilon^2 = 1 - 2 \cdot \frac{p}{r} + \frac{v^2}{\rho^2} \tag{24}$$

(24) shows clearly that ε is a pure number. p and r both are lengths, v and ρ both are velocities.