From Kepler via Hamilton to Newton

A very elegant version of “3 Kepler yield 1 Newton”

1. Kepler’s First Law
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3. Velocity $\vec{v}$ and the Hodograph
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$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}\varphi} = \frac{2 \cdot c}{p} \cdot \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix}$$

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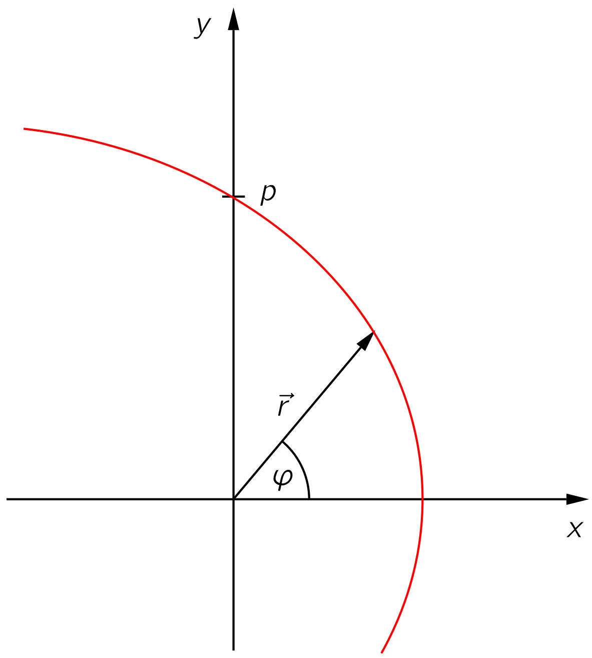
1 Kepler's First Law

K1: "The orbits of planets are ellipses, with the sun in one of their focal points."

The following mathematical formulation is slightly more general:

$$r(\varphi) = \frac{p}{1 + \varepsilon \cdot \cos\varphi} \tag{1}$$

This equation of a conic section in polar coordinates allows also parabolic and hyperbolic trajectories. The sun rests at the centerpoint of the coordinate system:



Differentiating (1) with respect to φ yields

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{p \cdot \varepsilon \cdot \sin\varphi}{(1 + \varepsilon \cdot \cos\varphi)^2} = \frac{\varepsilon}{p} \cdot r^2 \cdot \sin\varphi \tag{2}$$

The position $\vec{r}$ of the planet is given by

$$\vec{r} = r \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \tag{3}$$

Differentiating (3) with respect to φ using (2) and the product rule yields

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}\varphi} = \frac{\varepsilon}{p} \cdot r^2 \cdot \sin\varphi \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} + r \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \tag{4}$$

So much for the position vector $\vec{r}$.

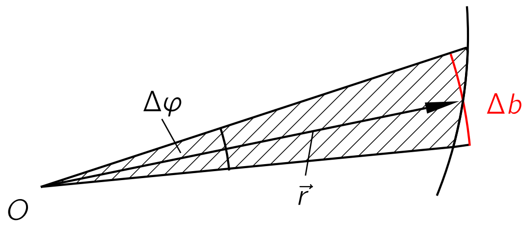
2 Kepler's Second Law

K2: "The position vector of a planet sweeps out equal areas in equal times."

For the mathematician this reads as

$$\frac{\mathrm{d}A}{\mathrm{d}t} = c = \text{constant} \tag{5}$$

Lets denote the area sweapt out by the position vector $\vec{r}$ with A. It corresponds to the hatched area in the following figure:



$$\Delta A = \tfrac{1}{2} \cdot r \cdot \Delta b = \tfrac{1}{2} \cdot r^2 \cdot \Delta\varphi$$

Using the chain rule we learn from (5) and this figure

$$c = \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}\varphi} \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{2} \cdot r^2 \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

This gives us the equation that will be of central importance throughout this paper:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{2 \cdot c}{r^2} \tag{6}$$

Using the chain rule we will replace all derivations with respect to time by derivations with respect to φ, multiplied by the right side of (6). This simplifies all the calculations enormously, compare for example with Kepler_03.pdf.

The idea to this approach comes from the beautyful paper "Von den Hüllkurvenkonstruktionen der Kegelschnitte zu den Planetenbahnen" , written by Erich Ch. Wittmann and published in 'Mathematische Semesterberichte' (2015) 62: p.17-35, Springer Verlag.

3 Velocity $\vec{v}$ and the Hodograph

For the velocity $\vec{v}$ we get from (6)

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}\varphi} \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{2 \cdot c}{r^2} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}\varphi}$$

Using (4), (1) and the Pythagorean theorem for the trigonometric functions we find

$$\begin{aligned}
\vec{v} &= \frac{2 \cdot c}{p} \cdot \varepsilon \cdot \sin\varphi \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} + \frac{2 \cdot c}{r} \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \\
&= \frac{2 \cdot c}{p} \cdot \varepsilon \cdot \sin\varphi \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} + \frac{2 \cdot c}{p} \cdot (1 + \varepsilon \cdot \cos\varphi) \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \\
&= \frac{2 \cdot c}{p} \cdot \varepsilon \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{2 \cdot c}{p} \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \qquad (7)
\end{aligned}$$

We define

$$\rho = \frac{2 \cdot c}{p} \quad \text{und} \quad h = \frac{2 \cdot c}{p} \cdot \varepsilon = \rho \cdot \varepsilon \qquad (8)$$

Rewriting (7) using (8) we get

$$\vec{v} = \begin{pmatrix} 0 \\ h \end{pmatrix} + \rho \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \qquad (9)$$

(9) is the equation of a circle with centerpoint $(0/h)$ and radius ρ! As a side result we have proven Hamilton's theorem about the hodograph of planetary motion in the gravitational field of the sun. The theorem follows from **K1** and **K2**.

In section 7 of Kepler_10.pdf you find pictures of the hodographs for all three cases: $\varepsilon < 1$ (ellipse), $\varepsilon = 1$ (parabola) and $\varepsilon > 1$ (hyperbola).

In the next section we need the derivative of $\vec{v}$ with respect to φ. From (7) we get immediately

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}\varphi} = \frac{2 \cdot c}{p} \cdot \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix} \qquad (10)$$

4 Acceleration $\vec{a}$

Again, we avoid the derivation with respect to time. Using the chain rule we get from (6) and (10)

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}\varphi} \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{2 \cdot c}{p} \cdot \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix} \cdot \frac{2 \cdot c}{r^2}$$

or

$$\vec{a} = \frac{4 \cdot c^2}{p} \cdot \frac{1}{r^2} \cdot \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix} = -\frac{4 \cdot c^2}{p} \cdot \frac{1}{r^3} \cdot \vec{r} \tag{11}$$

Obviously, vector $\vec{a}$ is antiparallel to vector $\vec{r}$. The acceleration is allways directed to the center of the coordinate system, that is to the sun. The acting force is a central force.

The absolute value of $\vec{a}$ is given by

$$|\vec{a}| = \frac{4 \cdot c^2}{p} \cdot \frac{1}{r^2} \tag{12}$$

The values of c and p differ from planet to planet. So the central force can follow the inverse square law only if the term $\frac{4 \cdot c^2}{p}$ has the same value for all planets. As we will see, this is exactly the statement of Kepler's third law !

5 Kepler's Third Law

K3: "The proportions of the squares of the orbital periods equal the proportions of the cubes of the semi-major axes."

Integrating (5) along a complete period T gives the area of the planet's ellipse:

$$c \cdot T = \pi \cdot a \cdot b$$

with a and b denoting the major and the minor semi-axes of the ellipse. Squared we have

$$c^2 \cdot T^2 = \pi^2 \cdot a^2 \cdot b^2$$

Substituting b^2 by $a \cdot p$ we get

$$c^2 \cdot T^2 = \pi^2 \cdot a^3 \cdot p$$

and hence

$$\frac{4 \cdot c^2}{p} \cdot T^2 = 4 \cdot \pi^2 \cdot a^3$$

From this we see that **K3** only holds if the term $\frac{4 \cdot c^2}{p}$ has the same value for all planets.

So for all planets we have

$$|\vec{a}| = \frac{4 \cdot c^2}{p} \cdot \frac{1}{r^2} = \frac{k}{r^2} \tag{13}$$

k takes the same value for all planets, it is a universal constant in our planetary system.

6 From Acceleration to Newton's Law of Gravitation

Now we know that each planet experiences an acceleration directed to the center of the sun given by the term

$$|\vec{a}| = \frac{k}{r^2}$$

where k has the same value for all planets. Two further steps lead from this to Newton's general law of gravitation:

First, the notion of force has to be *defined*. Here we use the simple version

$$\vec{F} = m \cdot \vec{a}$$

The force acting on a planet is given by its mass multiplied by acceleration.

Second, we have to take Newton's Third Law into account: All forces are *interactions* between bodies, abbreviated oftly by *actio = reactio*. From this deep finding of Newton follows that the mass of the sun has to be part of the formula for the attracting force, too. In addition, the problem needs a refined analysis, because also the sun has to move around the common center of mass. This actually leeds to a small correction of **K3**. For a complete derivation of all the details consult Kepler_01.pdf, sections 10 to 12.

Now we get Newton's general law of gravitation:

$$\vec{F} = m \cdot \vec{a} = m \cdot \frac{k}{r^2} \cdot \frac{-\vec{r}}{r} = m \cdot \frac{G \cdot M}{r^2} \cdot \frac{-\vec{r}}{r} = G \cdot \frac{M \cdot m}{r^2} \cdot \frac{-\vec{r}}{r} \tag{14}$$

For each planet we have

$$\frac{4 \cdot c^2}{p} = k = G \cdot M \tag{15}$$

with G standing for the universal constant of gravitation and M denoting the mass of the sun.

Kepler came close to this. He had a clear idea on a general attractive force between massive bodies. In the preface to his "New Astronomy" we read

> "Two stones, put to any place in the world, close to each other but far from any other similar body, will approach each other and meet at a place in between, quite alike two magnets. Each one will approach the other by a distance proportional to the mass of the other."

The mutual attraction of earth and moon was a given fact to him, and he saw in it the cause of the tides.

Newton was standing on the shoulders of giants, indeed.